TESE DE DOUTORAMENTO

ENTROPY IN HIGHER-CURVATURE THEORIES OF GRAVITY

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D./Dna. **Alejandro Vilar López**

Título da tese: **Entropy in higher-curvature theories of gravity**

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ENTROPY IN HIGHER-CURVATURE THEORIES OF GRAVITY

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En Santiago de Compostela, 21 de xullo de 2021

¿Qué cosa fuera corazón, qué cosa fuera?
¿Qué cosa fuera la maza sin cantera?
Un testaferro del traidor de los aplausos
Un servidor de pasado en copa nueva
Un eternizador de dioses del ocaso
Júbilo hervido con trapo y lentejuela
¿Qué cosa fuera corazón, qué cosa fuera?
¿Qué cosa fuera la maza sin cantera?

Silvio Rodríguez, *La maza.*

A mamá. A papá.
Á familia.

Agradecementos

Penso:

que rematou unha etapa máis; e non tería chegado ata aquí de non ser por unha boa chea de xente que foi fundamental para facelo. Primordialmente Jose, por suposto: con el comezou a miña historia na física teórica, e con el seguiu ata o día de hoxe. Grazas por facerme voar fóra cando foi o momento, e grazas tamén pola acollida de volta cando eu a busquei. Ter un director co que podes falar de ti a ti e co que compartes non só gusto pola física, senón tamén outra boa morea de intereses, é un luxo. Grazas tamén por poñer cordura e experiencia cando a min me faltou, a última vez na propia redacción da tese.

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un puido ter no ano máis raro das nosas vidas. E, por suposto, grazas á xente coa que compartín o día a día: a Rosalía, proba indubidable de que un cambio de compañeiros de piso a cegas pode saír mellor que ben; e ao mellor que me deu ter estudado física nesta cidade: Pedro, Adolfo, Pablo e Parga. Que nos quiten o bailado.

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Finalmente, á familia. Cantos máis anos cumpro máis entendo o significado desta palabra: o inmutable aínda cando o tempo non perdoa e impón que cambie, o incondicionalmente certo nun mundo cheo de incerteza. Incluír a toda a familia aquí sería imposible, pero non podo non nomear a quen verdadeiramente fixo posible que hoxe estea escribindo estas liñas, apoiándome durante os últimos 27 (case 28) anos. Aos padriños, que foron segundos pais (senón máis) en Compostela. A Mary, a Noe, a Adri, a Suso. Á enana de Mar; foi un pracer estar en Compostela estes catro anos véndote medrar. A Andre, por encherme de orgullo de irmán con absolutamente todos e cada un dos pasos que dá. A papá e a mamá, polo apoio incondicional e a certeza de que ninguén dará nunca un mellor consello ca eles. Por último, grazas tamén a quen, podendo verme comezar, non puido verme rematar a tese, sabendo que de seguro estaría orgulloso. E tamén ás que, podendo verme, o *cabrón* do tempo non llelo vai deixar desfrutar. Nada me fai sentir máis orgulloso que saber de onde veño.

Contents

Introduction

This thesis intends to collect and unify the results obtained over the past few years in collaboration with my supervisor, José Edelstein, and some other authors: Kostas Sfetsos, Aníbal Sierra-García, Riccardo Borsato, Linus Wulff, Elena Cáceres, Rodrigo Castillo Vásquez, Pablo Bueno, and Joan Camps [1–5].[1] They all fall under the somewhat broad umbrella of concepts represented in the title: entropy in higher-curvature theories of gravity. However, once we go into detail, things will become highly technical pretty fast and, in a clear manifestation of a common phenomenon in almost any scientific endeavor nowadays, it will be easy to lose sight of the big picture. This short introduction tries to remedy this. Being this a work on entropy in higher-curvature theories of gravity, it is probably a good idea to start asking ourselves what is entropy and why it is important, why are higher-curvature theories of gravity relevant, and how do these concepts relate to each other.

Entropy

The fundamental ideas and principles underlying thermodynamics and statistical mechanics have shown themselves robust enough to stand the test of time and, quite frequently, they have provided a guiding principle to uncover new and surprising laws of nature. Among them, entropy and the second law stand out on their own as part of the fundamental concepts. The second law emerged from the need to put on solid theoretical grounds many scattered ideas and empirical knowledge that were gained during the development of the first steam engines.[2] During the first half of the nineteenth century, Sadi

[1] While these works were written, two other papers were also published by the author, [6, 7]. They are not included in the present thesis because they fall in a somewhat different line of research, centered on cosmological applications of higher-curvature gravities.

[2] I hope the reader forgives the small historical digression. In words of Steven Weinberg [8]: "*learn something about the history of science, or at a minimum the history of your own branch of science. The least important reason for this is that the history may actually be of some use to you in your own scientific work. (...) the history of science can make your work seem more worthwhile to you. As a scientist, you're probably not going to get rich. Your friends and relatives probably won't understand what you're doing. And if you work in a field like elementary particle physics, you won't even have the satisfaction of doing something that is immediately useful. But you can get great satisfaction by recognizing that your work in science is a part of history*". So, for historical amusement, let me also recommend [9, 10]. And let the previous words be my humble tribute to one of the greatest figures in contemporary theoretical physics, sadly deceased while I was finishing this thesis.

Carnot, and later Émile Clapeyron, were the two key figures that realized the importance of reversibility in order to understand in an abstract way how efficient the engines that were being developed in the peak of the industrial revolution could become. Their way of reasoning would led Rudolf Clausius, many years later, to propose a definition of a new thermodynamic quantity that would be a signal of reversibility. This was the birth of entropy, which at the time was defined by connecting two states A and B through a reversible process as:

$$S(B) = S(A) + \int_A^B \frac{\delta Q_{\text{rev}}}{T} \ , \tag{1}$$

with δQ_{rev} measuring the amount of heat exchanged in a reversible way. Completely isolated systems can only evolve in the direction of increasing entropy. This introduces a notion of which processes – compatible with other thermodynamic principles as conservation of energy – are possible, thus providing the connection with the discussions about maximum attainable efficiency the more practically-minded scientists at the time were worried with.

The first big turn of this story came some years later, around the 1870s. Kinetic theory and the search for mechanical laws that governed the behaviour of matter started to convince some scientists that the second law could only be of statistical nature. This is a major change of paradigm. It was guided by Maxwell and, especially, by Ludwig Boltzmann, who in a series of papers developed the ideas that nowadays are taught in any statistical mechanics course around the world and which provide a microscopic interpretation of entropy. The essential idea can be summarized as follows, using somewhat more modern terminology and notation. Any macroscopic state of a system (characterized by macroscopic variables) corresponds to many detailed microscopic states of its fundamental constituents, each of them with a certain probability p_i of occuring by means of fluctuations. The entropy of the macroscopic state is:

$$S = -k_B \sum_i p_i \log p_i \ , \tag{2}$$

where k_B is Boltzmann's constant, which we shall omit from now on by working in appropriate units. If there are Ω equally likely compatible microstates with a given macrostate, then $p_i = 1/\Omega$ and we get the expression

$$S = \log \Omega \ , \tag{3}$$

which realizes the popular notion that entropy counts the (logarithm of the) number of microstates compatible with a given macrostate.

All this shows that entropy is a fundamental notion when going from a macroscopic theory (as it was the thermodynamical theory of gases at the time) to a corresponding microscopic model from which the large scale physics emerges. It is certainly surprising that an idea devised to understand in an abstract way practical machines is so closely tied to the fundamental physical theories underlying the microscopic structure of matter. The astonishment is probably only surpassed by the realization, about a century later, that entropy is a concept naturally built-in in our most successful theory of gravity: Einstein's General Relativity.

Black hole entropy

In a somewhat provocative spirit, we could say that General Relativity is a thermodynamic theory of gravity. This was not realized until many years after its original formulation, and we can speculate that Einstein himself would be pretty amazed to know so, since he was known to be quite fond of thermodynamics and statistical mechanics. The first clue towards this fact was in the form of mechanical laws obeyed by black holes [11], which are the most impressive macroscopic beasts predicted by General Relativity.[3] These were relations formally similar to the ones of thermodynamics, but people refused to interpret them as if they truly were so. After all, considering a black hole – which, by definition in the classical theory, is only capable of swallowing matter and is totally characterized by a few numbers giving its conserved charges – a proper thermodynamic object seemed at least questionable. Bekenstein was the first one to push this analogy further [14], motivated by considerations regarding violations of the second law in black hole spacetimes in case we do not assign a certain entropy to the black hole itself. Hawking's calculation of the quantum thermal emission of particles by black holes [15] was then the final result needed to attribute proper thermodynamic nature to them. Since then, we think of (equilibrium) black holes within General Relativity as objects with a temperature and an entropy given by:

$$T_H = \frac{\kappa}{2\pi}\,, \qquad S_{BH} = \frac{A}{4G_N} = \frac{A}{4\ell_P^2}\,, \tag{4}$$

with κ the surface gravity of the black hole horizon, A its area, and ℓ_P the Planck length – we work in 4 dimensions in this discussion.

The question naturally arises when linking black hole entropy with our previous historical introduction: if black holes have entropy, what are the microscopic states that provide the statistical interpretation? This question, in one form or another, has been around since the mid-70s [16]. It is fair to say that we do not yet have a satisfactory answer – apart from some very special and controlled situations within string theory, [17]. Since microscopic physics laws fall into the general framework of quantum mechanics, it is in fact expected that a full answer to the question of what are the black hole microstates would necessarily lead to a fully consistent quantum theory of gravity, thus providing a solution to the problem that has occupied the minds of many theoretical physicists since about a century ago. Black hole entropy might well be the gateway to quantum gravity, and therefore it is certainly a matter to which a good deal of attention must be devoted.

Holography and entanglement entropy

One of the significant properties of the Bekenstein-Hawking formula for black hole entropy is the fact that it is proportional to the horizon area. Since entropy is a measure of the number of degrees of freedom,[4] in a conventional local quantum field theory it typically scales with the volume of a given region. The weird area-law behaviour of black hole entropy led to some speculations about the *holographic* character of gravity, [18, 19]. By this it is meant that the description of gravity in a certain region in terms of microscopic

[3] And, since some years ago, observed experimentally [12, 13].

[4] This can be seen via an order of magnitude estimate from the previous statistical definition of entropy, (3). Assume N degrees of freedom each of which can be in one of s states, then the total number of states is s^N, and $S = \log s^N = N \log s \sim N$.

degrees of freedom is perhaps to be done by means of some kind of local theory living on the boundary of the region. Locality and geometric notions in the interior – fundamental building blocks of the classical description of gravity – would then be emergent concepts. This is only a very vague description of the ideas underlying the holographic principle, the reader is encouraged to consult one of the many sources available in the literature to get a more precise picture of the subject. A very nice review can be found in [20].

The most important concrete realization of the holographic principle we have so far is the AdS/CFT correspondence [21–23], as the huge amount of research it has sparked shows. This is by now a vast subject, with many ramifications and applications, so we shall restrict ourselves to explaining the main ideas needed to put into context the work presented in this thesis. Further information, with different focuses, can be found in one of the many available reviews, among which we highlight [24–28]. In the strongest version of the original formulation, AdS/CFT proposes a dynamical equivalence between a certain (purportedly) quantum gravity theory and a conformal field theory:

4-dimensional $\mathcal{N}=4$ SU(N) Super Yang-Mills (SYM) theory	$\Leftrightarrow$	Type IIB superstring theory on $\mathrm{AdS}_5\times\mathbb{S}^5$

On the left hand side, we have a conformally invariant field theory without gravity in four dimensions. It has $\mathcal{N}=4$ supersymmetry, and SU(N) as the gauge group. The free parameters are N itself, and the gauge coupling g_{YM}. On the right hand side, we have a superstring theory defined on $\mathrm{AdS}_5\times\mathbb{S}^5$.[5] String theory is built starting from extended fundamental objects (strings) which possess in their spectrum of quantum excitations a tower of particles with different spins. In particular, closed strings have spin-2 symmetric, traceless, massless excitations, which are identified as gravitons. Thus, it is hoped that it can provide a consistent quantum theory of gravity. In the previous correspondence, the free parameters on the string theory side are the ratio $L/\sqrt{\alpha'}$ between the curvature radius L of the AdS_5 and $\mathbb{S}^5$ spaces (which are equal) and the string length $\ell_{\mathrm{s}}=\sqrt{\alpha'}$; as well as the string coupling constant controlling the loop expansion g_{s}. The relation between the parameters in the two sides of the correspondence is:

$$g_{\mathrm{YM}}^2 = 2\pi g_{\mathrm{s}}\ , \qquad 2\lambda = 2g_{\mathrm{YM}}^2 N = L^4/\alpha'^2\ , \tag{5}$$

where we introduced the 't Hooft coupling $\lambda\equiv g_{\mathrm{YM}}^2 N$ in the field theory side. Apart from these relations among the parameters of the theories, the basic identification that would relate CFT and string theory quantities is proposed to be, schematically:

$$\left\langle \exp\left(\int \phi_0(x)\mathcal{O}(x)\right)\right\rangle_{\mathrm{CFT}} = \mathcal{Z}_{\mathrm{string}}\Big|_{\phi(z,x)\underset{z\to 0}{\sim} z^{\Delta-d}\phi_0(x)}\ , \tag{6}$$

where ϕ denotes any field of the string theory, which is integrated over in the string partition function but restricted to behave as $\lim_{z\to0} z^{d-\Delta}\phi(z,x)\to\phi_0(x)$ when approaching the AdS boundary at $z\to 0$. In the dual CFT, the boundary value $\phi_0(x)$ acts as a source in the generating functional of correlation functions for the operator $\mathcal{O}$, which has dimension Δ.

[5]Since the field theory is 4-dimensional, holography would suggest it describes a 5-dimensional quantum gravity theory, but in this discussion we seem to get 10 dimensions. The 5 compact dimensions of the sphere should be reduced via a Kaluza-Klein procedure, obtaining in this way a five-dimensional gravitational theory.

In reality, we do not have control of the full non-perturbative string theory ($\mathcal{Z}_{\text{string}}$ is not known), so the previous duality is more of a general philosophy than an actual equivalence between two well understood sides. However, things improve after some limits are taken, rendering the situation more under control. In particular, we want to supress loop stringy contributions, so we send $g_{\text{s}} \to 0$, which correspondingly sets $g_{\text{YM}} \to 0$ in the field theory. The product $\lambda = g_{\text{YM}}^2 N$ is kept constant in this first step and given by the ratio L^4/α'^2, so the field theory must have a large number of degrees of freedom, $N \to \infty$. After this large-N limit is taken, we can go to the strongly coupled field theory regime, $\lambda \to \infty$, in which the extended character of strings becomes irrelevant, $\alpha'/L^2 \to 0$. Thus, the string theory becomes classical (super)gravity, were many calculations can be explicitly performed. It is in this setup where most checks of the AdS/CFT correspondence have been done, exploiting also some of the special properties the supersymmetric field theory has, which allow to compute quantities at weak coupling and show that they can be properly extrapolated to strong coupling. Some examples of results that match between the two sides of the duality when computed in the large-N and strongly coupled CFT limit are correlation functions [29], or conformal anomaly coefficients [30]. We can also write the more controlled version of (6) after the previously mentioned limits are taken. It reads:

$$\left\langle \exp\left(\int \phi_0(x)\mathcal{O}(x)\right)\right\rangle_{\text{CFT}}\Bigg|_{N\to\infty,\lambda\to\infty} = e^{-\mathcal{I}_{E,\text{SUGRA}}[\phi_0]} \,, \tag{7}$$

where now the right-hand side has been evaluated in the saddle-point approximation, which produces the on-shell action of the supergravity theory which constitutes the low-energy limit of string theory. The geometry is still AdS_5 $(\times\mathbb{S}^5)$, the boundary behaviour of ϕ is as before, and we write subscript E in the action to indicate that we work in Euclidean signature.

The previous two paragraphs were quite technical, and in fact their only purpose is to motivate the less rigorous version of AdS/CFT we will introduce now. The duality between $\mathcal{N} = 4$ SYM and type IIB superstring theory in the controlled large-N and strong coupling limits has given us quite a few reasons to believe that gravitational theories in an AdS background can be thought to be dual to a certain conformal field theory in the boundary. This is so even if we do not know exactly what is the particular field theory we are dealing with. Gravity with AdS boundary conditions provides a natural "box" in which we can place quantum fields (including gravitons), and this defines a certain CFT in the boundary. Naturally, this will not be *any* CFT, some particular properties must be present in order to come from a gravitational dual. In general, just like in the concrete realization with $\mathcal{N} = 4$ SYM, we expect to have a large number of degrees of freedom and to be in a strongly coupled regime. It has been a huge leap to go from the technically convoluted presentation of AdS/CFT above (5) to this vague proposal, so in case the reader feels the need of a more thorough discussion regarding this issue, [28] provides a nice starting point. From now on, we will content ourselves with a relation like (7), where the on-shell bulk action can be derived from any (well-behaved) gravitational theory with AdS boundary conditions.

After this long exposition of the basic idea behind AdS/CFT, let us return now to entropy and see how it fits within the framework of holography. The basic object to make the connection is the von Neumann entropy. Let ρ be the density matrix characterizing a certain quantum state, we define its von Neumann entropy as:

$$S(\rho) = -\mathrm{Tr}\ (\rho \log \rho)\ . \tag{8}$$

This formula is the natural extension to the quantum realm of the classical statistical entropy, (2). If ρ is a statistical ensemble of a certain set of orthonormal states $\{|\psi_i\rangle\}$ with probabilities p_i, then the previous expression exactly reduces to (2). We are particularly interested in the case where ρ is obtained from a pure state by tracing out a subset of the degrees of freedom. By this we mean that we take $|\psi\rangle$ to be a state defined in some product Hilbert space, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, and we define $\rho = \rho_A$ as:

$$\rho_A = \mathrm{Tr}_{\bar{A}}\, (|\psi\rangle\langle\psi|)\ . \tag{9}$$

In this case, $S_{\mathrm{EE}}(A) \equiv S(\rho_A)$ is known as the entanglement entropy of A. It measures the amount of ignorance we have about the state of subsystem A even when we know the full state to be $|\psi\rangle$. Notice that this is an intrinsically quantum phenomenon, since classically knowing the state of a system means knowing the state of its constituents.

The previous construction can be done in a quantum field theory, where $|\psi\rangle$ is a certain state and A is taken to be a particlar region of a spatial slice. In this case, we would be tracing out the degrees of freedom outside A, and $S_{\mathrm{EE}}(A)$ would be the entanglement entropy of region A in the state $|\psi\rangle$.[6] This puts us in a position to make contact with AdS/CFT. We can ask ourselves: how is the entanglement entropy of a region A of the CFT to be calculated from the gravitational side? The answer, found by Ryu and Takayanagi [32, 33] when the bulk gravity is described by the Einstein-Hilbert Lagrangian, is extremely simple and surprising – in part due to its simplicity; since entanglement entropy in field theory is a quantity extremely difficult to calculate, while the corresponding holographic recipe could be followed by any sophomore physics student. The Ryu-Takayanagi proposal instructs us to consider region A as part of the AdS boundary in which the field theory lives. Then, we consider surfaces penetrating into the bulk which end on the boundary of region A, and we find the one among them which has minimal area. Call this surface Γ_A; the entanglement entropy of A is then given by:

$$S_{\mathrm{EE}}(A) = \frac{\mathrm{Area}(\Gamma_A)}{4G_N}\ . \tag{10}$$

So we have turned a complicated QFT problem into a simple question of minimizing a geometric quantity. This is the beauty of holography in its finest form!

At this point, one might ask why should we care at all about entanglement entropy of CFT regions. After all, we argued that black hole entropy could be a gateway towards quantum gravity, but entanglement entropy, even if the Ryu-Takayanagi prescription is a nice result, could be little more than a mathematical characterization of some field theory property of questionable interest. There are many reasons for this to be false, starting from the fact that entanglement entropy is a powerful tool widely used in condensed matter systems to characterize field theory states. In fact, we will use it in this thesis as a way to extract information about the CFT a given bulk gravitational theory defines. To

[6]Doing this in a QFT setup is actually tricky. The continuous nature of field theories complicates the process of separating the degrees of freedom between A and $\bar{A}$. See [31] for a (very formal) treatment of this issue.

make contact with our previous discussions, however, it is useful to exploit the striking similarity between (10) and the black hole entropy formula, (4). If we consider a black hole spacetime in the bulk, and apply (10) to the full boundary state of the CFT, we find that the surface Γ_A does in fact coincide with a spatial section of the horizon, so that the black hole entropy coincides with the von Neumann entropy of the CFT state. Black holes are dual to thermal states, so the thermal entropy of the CFT coincides with the black hole entropy in the bulk. This provides a first hint towards a surprising fact shown by [34] while giving arguments supporting the validity of (10): if we assume gravity is holographic, there is a notion of generalized gravitational entropy which can always be understood in terms of the von Neumann entropy of dual field theory states. So, in the end, black hole entropy and (holographic) entanglement entropy are closely related. This should be more than enough to motivate the study of entanglement entropy in the context of holography.

Higher-curvature theories of gravity

Let us conclude this short introduction by focusing now in the other guiding principle of the thesis: higher-curvature theories of gravity. Most of the results we presented up until now are only valid when the gravitational theory at hand is Einstein gravity. That means our action in D dimensions takes the form:

$$\mathcal{I}_{\rm EH} = \frac{1}{16\pi G_N}\int \mathrm{d}^D x\sqrt{|G|}\,(R-2\Lambda)+\mathcal{I}_{\rm mat}\ , \tag{11}$$

where $\mathcal{I}_{\rm mat}$ includes extra matter fields, which may or may not appear. The need to include higher powers of the curvature tensor in the previous action can be understood from different perspectives. The most direct one probably comes from adopting a Wilsonian perspective and considering the Einstein-Hilbert action a low-energy effective theory, which should receive perturbative corrections weighted by coefficients with inverse mass dimension coming from a legitimate UV-completion. That means our gravitational action would rather look like:

$$\mathcal{I}_{\rm grav} = \frac{1}{16\pi G_N}\int \mathrm{d}^D x\sqrt{|G|}\left(R-2\Lambda+\frac{\beta_2}{M_\star^2}\mathcal{R}^{(2)}+\frac{\beta_3}{M_\star^4}\mathcal{R}^{(3)}+\dots\right)\ , \tag{12}$$

where we are only writing the schematic form at each order, so $\mathcal{R}^{(k)}$ are actually several k-th order powers of curvature tensors contracted in different ways. The scale $M_\star$ determines when this corrections start to become relevant (we take the couplings to be $\beta_k\sim\mathcal{O}(1)$), for processess with energies way below $M_\star$, the leading order Einstein-Hilbert theory correctly reproduces the physics. This situation is actually predicted by our (arguably) best candidate for a quantum theory of gravity: string theory. The low-energy dynamics of string theory is governed by an effective gravitational action with some extra fields, and higher order corrections appear weighted by the string length, $M_\star^{-2}\sim\alpha'$ [35, 36].

There are other reasons to consider higher-curvature contributions to the gravitational Lagrangian. It has been known for some time now that the inclusion of higher-curvature terms in the gravitational action can cure the non-renormalizability of the Einstein-Hilbert theory [37], if only at the cost of introducing ghostly states in the spectrum. Furthermore, some higher-curvature terms can provide useful phenomenological models in different contexts, especially notable are those arising in cosmology, such as the famous Starobinsky R^2 model [38, 39]. We could add to this list, in the spirit of the previous discussion regarding AdS/CFT, the interest of higher-curvature theories of gravity as "phenomenological" holographic models. By this we mean that, if we take a well-behaved[7] bulk gravitational theory with AdS asymptotics as being dual to some unknown boundary CFT, the inclusion of higher-curvature terms can allow us to access field theories with properties which are impossible to obtain by means of Einstein gravity in the bulk. The archetypical example of this is the fact that 5-dimensional AdS Einstein gravity is known to be dual to a CFT with equal trace anomaly coefficients, $a = c$. Adding higher-curvature terms we obtain theories for which this ceases to be true. Another significant example is the KSS bound [44] on the ratio between shear viscosity and entropy density. The conjectured lower bound, provided by Einstein gravity, was called into question after the inclusion of higher-curvature corrections [45–47]. When discussing holographic entanglement entropy, we will use higher-curvature gravities in this spirit, as probes of different CFTs.

[7]By well-behaved we can mean many different things, and in fact it is probably impossible to provide an exhaustive list of all the requirements needed. In general, we have in mind a bulk theory which has all the properties needed to provide a properly defined dual CFT. Frequently, key properties of the CFT such as positivity of energy have a counterpart in the gravity picture, in this cause causality [40–42]. Imposing causality does in fact seem to restrict a lot the set of healthy higher-curvature gravitational theories [43], and it is very likely that only in certain perturbative regimes we can make sense out of them.

Outline of the thesis

This thesis is structured in two parts. Their common connection is the fact that we will always be dealing with higher-curvature theories of gravity, and particularly with some notion of entropy within them.

In the first part, we will study black hole entropy in a two parameter family of theories presented by Marqués and Núñez [48], building up on earlier work by Hohm, Siegel and Zwiebach [49, 50]. Introducing these theories is the primary goal of chapter 1. Their main property is that they constitute a generalization of the first order in α' low-energy string effective theories, which preserve one of their characteristic attributes: T-duality invariance. After reviewing the leading order low-energy string effective actions and the T-duality rules, we introduce the first-order corrected counterparts of Marqués and Núñez, which for some values of the parameters reduce to string theory actions, while for others do not. We write the corrected T-duality rules, and we consider some general first-order redefinitions of the fields appearing in the theory, obtaining also the corresponding rules for the redefined fields. These results, presented in [3], help to make the connection with previously known expressions for T-duality rules in some two-loop low-energy string effective theories.

Chapter 2 studies how black hole entropy is to be calculated in the previous family of theories. It is based on [2], and after a quick review of Wald's method to compute black hole entropy as a Noether charge associated with diffeomorphism invariance [51], we discuss the subtleties present in the family of theories of Marqués and Núñez, and show how they are to be dealt with. The outcome of this procedure is a closed expression for black hole entropy, the technical details needed in order to obtain it are relegated to appendix A. Within the same chapter, we consider a general bifurcate Killing horizon, and by means of a suitable set of coordinates, and using the previously obtained form for the black hole entropy, we prove that the horizon entropy is invariant under the corrected T-duality rules of the previous chapter. As a byproduct of this result, in the process we prove that the temperature is also invariant, thus showing that, for any values of the two parameters characterizing the theories, T-duality preserves black hole thermodynamic quantities.

Part I concludes with chapter 3, which is a concrete example that demonstrates the invariance of entropy and temperature starting from an explicit solution of the two-parameter family of theories. This solution is a slight modification of the three-dimensional BTZ black hole. After presenting this background, we explicitly compute its temperature and entropy. Computing then its T-dual by means of the rules of chapter 1, we obtain a first-order corrected black string, which also has an horizon with the same temperature and entropy of the BTZ solution. These results were already presented in [1].

The second part of the thesis deals with holographic entanglement entropy computations when the bulk theory contains arbitrary higher-curvature terms built out of contractions of the Riemann tensor. Chapter 4 is a general discussion on how the holographic entanglement entropy functional is obtained in this situation, based on the works [4, 5]. After a review of the Lewkowycz-Maldacena construction [34], which is essential in deriving the functional for higher-curvature theories [52, 53], we present the so-called "splitting problem", which forces us to work in a regime in which the higher-curvature couplings are perturbative. In this setup, we develop a novel rewriting of the holographic entanglement entropy functional which presents clear advantages both at the conceptual and the technical level. The proof of the equivalence between the new form of the functional and the one previously known in the literature can be found in appendix B. Our improved expression allows us to discuss the general structure of the functional depending on the number of Riemann tensors in the Lagrangian, as well as to obtain the explicit form of the functionals for quadratic, cubic, and quartic theories. The most cumbersome expressions for the holographic entanglement entropy functionals of cubic and quartic theories are relegated to appendix C, in order to avoid clutter.

Chapter 5 puts into use the previously obtained functionals for theories up to cubic order in curvature tensors, following the results presented in [5]. In a pure AdS geometry, holographically dual to the vacuum state of a certain CFT, we compute the holographic entanglement entropies of a family of different boundary regions. The terms in these entropies which are independent of the UV regulator are known as universal terms, and they provide meaningful information about the CFT dual to a given gravitational theory. Including up to cubic corrections in the bulk allows to access different CFTs in the boundary, and therefore characterizing their properties is important. After a general presentation of the universal terms of entanglement entropy, we consider spheres, slabs, (hyper)cylinders, and corners in the boundary; and compute the corresponding holographic entanglement entropies by means of the associated bulk surface. In each case the universal terms obtained are discussed, matching when possible with results previously known in the literature, and emphasazing those which provide interesting new information. Of particular interest is the universal term for boundary corners in three dimensional CFTs, known as the corner function. Cubic corrections in the bulk allow to obtain a different corner function from the one Einstein gravity produces. This is the first example of an holographic corner function with this property, since quadratic corrections were shown in the past to produce a corner function proportional to the Einstein gravity one [54].

We conclude the thesis with a brief summary of the work done and some final conclusions and possible future directions in chapter 6. These complement and generalize the more detailed discussions presented at the end of each of the chapters.

Aims, objectives and methods

The main goal of this thesis is to elucidate how entropy is to be calculated in certain gravitational theories with higher-curvature terms in the action, and what are some of its most relevant properties. We can distinguish two guiding questions, each of them corresponding to one of the two parts of the text.

The first part considers theories in which T-duality is present as a symmetry at the level of classical solutions. The theories are not only low energy string effective actions – in which case we have a sigma model picture of the origin of T-duality, which instructs us to consider it a total physical equivalence –, but also a generalization of them. This means that for theories with no known sigma model origin, we do not know whether T-duality is a total equivalence or just a solution generating technique which does not necessarily preserve the physical properties of the backgrounds. Given that black hole entropy should be related to the microscopic properties of the theory, it appears as a suitable observable to distinguish between the two options. So, the main question is, given a black hole solution of the classical theory, does T-duality preserve its entropy, even for the cases which cannot be attributed a string sigma model origin? This guiding question poses some other problems on its own, such as how the entropy is to be calculated in the theories considered – a non-trivial issue in itself, due to the particular form of the local Lorentz symmetry present in the theories. These can be considered as auxiliary problems which need to be solved in order to answer the main question of the first part.

The second part deals with holographic entanglement entropy in the presence of higher order contractions of the Riemann tensor in the action. The functional computing such a quantity has been known for some years, but the particular procedure to obtain it in a given theory is cumbersome and badly suited for computer based derivations (usually the only reasonable method to deal with general higher-curvature gravities, due to its inherent technical complexity). The main question we want to answer is then: can we understand the process behind the derivation of the functional, to a point which allows us to simplify it and make it more amenable to both particular computations and general discussions? The resolution of this question in the affirmative will give us as a byproduct some extra problems, such as what is the general structure of the functional depending on the form of the Lagrangian, or which are the universal terms of the entanglement entropy of certain boundary regions when we include perturbative higher-curvature corrections in the bulk. Once again, these are to be regarded as follow-up investigations, motivated by the resolution of the main problem of this second part.

Methodology

This is a thesis in theoretical high-energy physics, and as such the main tools to solve the previously posed questions are a vast and always growing literature – as the large bibliography included in this thesis proves –, and the occasional help of computer software to carry out complicated calculations. In this second aspect, I must acknowledge the help provided by the tensor computer algebra package *xAct* [55], an add-on to Wolfram's *Mathematica*. Of course, it is also essential to learn and properly apply the mathematical technology needed to formulate and solve the questions, usually a good deal of differential geometry and analysis. Trying to discuss here all the details of these tools in a bird's-eye-view would certainly be in vain, so we relegate the presentation of the more technical and unfamiliar bits to the particular chapters where they are needed, and trust that if the reader is having a look at this thesis is because he or she has the necessary basic background.

Apart from that, as the reader surely knows if he or she has ever worked in this field, there is not a step by step procedure to solve the questions posed I can describe here. Usually the process goes something like this: you start with a more or less clear notion of the question you want to answer, you dig into the literature to see what previous results are relevant, you struggle to solve some important calculation to answer the question, and you finally communicate your results in the form of a paper. This is quite frequently a process that can go not only forward, but also backwards. Thus, while calculating things you might realize that the question you posed is somehow ill-defined and have to be reconsidered. Furthermore, it is fair to acknowledge that the process is normally enriched by discussions with your collaborators or other scientists working in the field, which can be of enormous value in order to decide which are the most interesting questions to pose, or which previous works that might have gone unnoticed to you are a good help to solve the problem you are currently facing. I think this is as much as I can honestly say about my methodology for writing this thesis.

Notation and conventions

This thesis will make extensive use of certain conventions, so for the sake of clarity we collect them here all together. This will also make them easier to consult. First of all, on a linguistic level, we will sometimes use abbreviations, especially to avoid large titles or repetitions. Hopefully these are self evident; some of the most frequent ones will be EE for entanglement entropy, HEE for holographic entanglement entropy, RT for Ryu-Takayanagi (surface), or BdR for the Bergshoeff-de Roo theory introduced in chapter 1. On the technical side, the first convention is a quite standard one in the theoretical high-energy community: we avoid carrying unnecesary constants by working in units with $c = \hbar = k_B = 1$. We will however write explicitly Newton's gravitational constant G_N.

Dimensions and indices

The spacetime in which gravitational theories live is assumed to be D-dimensional. Therefore, when discussing holographic entanglement entropy, the boundary theory is assumed to live in $D-1$ dimensions. When we want to emphasize the CFT dimensionality, as it will be the case in most holographic discussions, we define $d = D-1$ as the CFT dimension. Index notation will also be ubiquituous in this thesis, so let us establish it as clearly as possible from the beginning. Coordinate indices in D dimensions will be $M, N, \ldots$; while coordinate indices in $d = D-1$ dimensions will be $\mu, \nu, \ldots$. In the first part of the thesis we will also deal with flat indices, which are taken to be $A, B, \ldots$ in D dimensions, and $a, b, \ldots$ in $D-1$ dimensions.[8] The relation between coordinate and flat indices is, as usual, provided by the vielbein $E_M{}^A$:

$$G_{MN} = \eta_{AB} E_M{}^A E_M{}^B \ , \tag{13}$$

with η_{AB} the D-dimensional Minkowski metric. In the second part of the thesis, we will not deal with flat indices, but we will split the $D = d+1$ dimensions of the bulk into normal directions to the holographic entanglement entropy surface and tangential ones. Since this surface has codimension 2, there are two normal directions, indexed by $a, b, \ldots$; and $d-1$ tangential ones, indexed by $i, j, \ldots$. The two different meanings of lowercase latin indices should not be a problem, since they will be used in completely different contexts. Occasionally, some extra set of indices might be needed (*e.g.*, in the DFT

[8] Flat indices in $D-1$ dimensions appear in the first part of the thesis while dimensionally reducing along the T-duality symmetry direction. In chapter 1 the precise construction is discussed in detail.

discussion of chapter 1), but this will not be frequent and consequently we will clearly discuss the notation when it first appears.

Local Lorentz invariance and differential forms

As already mentioned, the first part of the thesis will make extensive use of a formalism in which local Lorentz invariance is manifest. We will therefore deal with the typical objects of the vielbein formalism, such as the spin connection,

$$\Omega_{MA}{}^{B} = E_{A}{}^{N}\partial_{M}E_{N}{}^{B} - \Gamma^{N}_{MP}E_{A}{}^{P}E_{N}{}^{B} \ , \tag{14}$$

its associated curvature,

$$R_{MNA}{}^{B} = \partial_{M}\Omega_{NA}{}^{B} - \partial_{N}\Omega_{MA}{}^{B} + \Omega_{MA}{}^{C}\Omega_{NC}{}^{B} - \Omega_{NA}{}^{C}\Omega_{MC}{}^{B} \ , \tag{15}$$

and the gravitational Chern-Simons form,

$$\Theta_{MNR} = \Omega_{[MA}{}^{B}\partial_{N}\Omega_{R]B}{}^{A} + \frac{2}{3}\Omega_{[MA}{}^{B}\Omega_{NB}{}^{C}\Omega_{R]C}{}^{A} \ , \tag{16}$$

where antisymmetrization only acts on curved indices. Sometimes we will also deal with torsionful connections, the correspondig objects are formally defined by the same expressions, but with the modified connection.

Another useful tool when discussing the entropy construction à la Wald is the language of differential forms. We will not make use of all of its power, but sometimes it will be of great help. Basic operations such as wedge product or exterior differentiation are defined as in most textbooks, see *e.g.* [56]. The Hodge dual that transfoms between p vectors and $(D-p)$-forms is written using a notation inspired by [57], which is particularly well suited for the black hole entropy discussions. Given a $(D-p)$-form $\mathbf{A}$, we can write it in terms of a p-vector (p antisymmetric upper indices) as:

$$\mathbf{A} = \frac{1}{(D-p)!}A_{M_1\ldots M_{D-p}}\mathrm{d}x^{M_1}\wedge\cdots\wedge\mathrm{d}x^{M_{D-p}} = A^{N_1\ldots N_p}\left(\mathrm{d}^{D-p}x\right)_{N_1\ldots N_p} \ , \tag{17}$$

where

$$\left(\mathrm{d}^{D-p}x\right)_{N_1\ldots N_p} \equiv \frac{1}{p!(D-p)!}\epsilon_{N_1\ldots N_p M_1\ldots M_{D-p}}\mathrm{d}x^{M_1}\wedge\cdots\wedge\mathrm{d}x^{M_{D-p}} \ , \tag{18}$$

with $\boldsymbol{\epsilon}$ the metric-induced volume form. The Hodge star of the form $\mathbf{A}$, written $\star\mathbf{A}$, is just the differential p-form obtained by lowering indices in $A^{N_1\ldots N_p}$. There are a couple of operations which are particularly simple in this notation. One is the interior product; for a D-form such as the Lagrangian $\mathbf{L} = \mathcal{L}\left(\mathrm{d}^{D}x\right)$:

$$i_{\zeta}\mathbf{L} = \mathcal{L}\,\zeta^{M}\left(\mathrm{d}^{D-1}x\right)_{M} \ , \tag{19}$$

for a certain vector field ζ. The other one is exterior differentiation:

$$\mathrm{d}\mathbf{A} = \mathrm{d}\left[A^{N_1\ldots N_p}\left(\mathrm{d}^{D-p}x\right)_{N_1\ldots N_p}\right] = \nabla_{R}A^{N_1\ldots N_{p-1}R}\left(\mathrm{d}^{D-p+1}x\right)_{N_1\ldots N_{p-1}} \ . \tag{20}$$

This simplifies a lot the discussion about conservation laws.

Two final comments about differential forms. First of all, we have been using boldface symbols for forms, which is useful in some cases to distinguish between a form and the dual vector (*e.g.*, the Lagrangian form $\mathbf{L}$ and the Lagrangian scalar $\mathcal{L}$). We will keep doing this when confusion might arise, but otherwise we do not do it and let the context decide when we are using differential forms. The other comment is very particular to discussions around black hole entropy. It can be shown [58] that if we have a bifurcate Killing horizon with bifurcation surface $\mathcal{B}$ (which is $(D-2)$-dimensional), the following identity holds:

$$\left(\mathrm{d}^{D-2}x\right)_{MN}\big|_{\mathcal{B}} = \frac{1}{2} n_{MN} \bar{\boldsymbol{\epsilon}} \ , \tag{21}$$

where n_{MN} is the binormal to $\mathcal{B}$ normalized as $n_{MN}n^{MN} = -2$, and $\bar{\boldsymbol{\epsilon}}$ is the induced volume form in $\mathcal{B}$.

Geometry of bulk entanglement surfaces

In discussions concerning holographic entanglement entropy, we will need to consider geometrical quantities characterizing the bulk codimension-2 RT surface. In these cases we work in Euclidean signature, and decompose the bulk metric as:

$$G_{MN} = h_{MN} + \delta_{ab} n^a{}_M n^b{}_N \ , \tag{22}$$

where $n^a{}_M$ for $a = 1, 2$ are two orthogonal unit normals to the surface. These define the binormal to the surface and the normal projector:

$$\epsilon_{MN} \equiv \epsilon_{ab} n^a{}_M n^b{}_N \ , \qquad \perp_{MN} \equiv \delta_{ab} n^a{}_M n^b{}_N \ . \tag{23}$$

In addition to intrinsic curvatures of the surface given by h_{MN}, we will have to deal also with extrinsic curvatures, defined as:

$$K^a{}_{MN} \equiv h^R_M h^S_N \nabla_R n^a{}_S \ , \qquad K^L{}_{MN} \equiv K^a{}_{MN} n_a{}^L \ , \tag{24}$$

where in the second expression we simply convert the normal index to a spacetime one, using $n_a{}^L \equiv \delta_{ab} n^b{}_M G^{ML}$. We will also use a shorthand notation for contraction of the indices of a tensor only along normal directions:

$$V^a_{\ a} = V^M{}_N \, n^a{}_M n_a{}^N = V^{MN} \perp_{MN} \ . \tag{25}$$

Sometimes it will be useful to take adapted coordinates to the surface, following popular conventions in the literature. We parametrie the normal directions in terms of complex coordinates $(z, \bar{z})$, while we set coordinates y^i for the surface, in such a way that:

$$\mathrm{d}s^2 = \mathrm{d}z \, \mathrm{d}\bar{z} + h_{ij} \mathrm{d}y^i \, \mathrm{d}y^j \ . \tag{26}$$

In these coordinates, the only non-vanishing components of the normal part of the metric are $G_{z\bar{z}} = G_{\bar{z}z} = 1/2$, and $G^{z\bar{z}} = G^{\bar{z}z} = 1/2$ for the inverse. It is clear from (24) that the components of the extrinsic curvature are non-vanishing only if the last pair of indices takes the form $K^a{}_{ij}$, and the contraction (25) is just $V^a_{\ a} = V^z_{\ z} + V^{\bar{z}}_{\ \bar{z}}$.

Other conventions

Some extra symbols and general conventions are the one for the Lie derivative with respect to a vector field V, $\mathcal{L}_V$, and that to indicate equality on-shell (*i.e.*, when the equations of motion are applied), $\cong$. This one is going to be used only to emphasize that the equations of motion are needed, something which is essential in some steps of Wald's construction to obtain black hole entropy. In any asymptotically AdS space, $L_\star$ will be the curvature radius of the spacetime. Notice that when higher-curvature terms are present, several curvature radii might provide solutions of the equations of motion. In particular, the AdS curvature radius does not have to coincide with the cosmological constant length scale appearing in the Lagrangian of the theory, as it is the case in Einstein gravity. We also use conventions for symmetrization and antisymmetrization normalized to 1, so that we divide by the factorial of the number of indices symmetrized or antisymmetrized (*e.g.*, $V_{[M} W_{N]} = (V_M W_N - V_N W_M)/2!$).

Part I

Black hole entropy, higher-curvature gravity and T-duality

1

T-duality invariant effective actions

String theory has become a milestone of high-energy theoretical physics during the past half-century. Originally born not as a theory of strings, but as a model to describe strong interactions [59, 60], many surprises were waiting along the development of the theory. The most radical one was, possibly, the fact that string theory places classical gravity, as described by the theory of General Relativity, in a consisten quantum-mechanical framework [61], becoming in this way a viable candidate for a model of quantum gravity. Despite this, it is fair to say that there are several aspects of string theory we do not yet fully understand, and it might well be the case that there are as many surprises waiting for us in future developments of the theory as there have been in the past. In the present chapter, though, we will not be concerned with all the particular details of string theory. We will only focus on the fact that its low-energy dynamics reproduces that of a gravitational theory, with some extra fields, and we will consider perturbative corrections to this low-energy theory. This will set the stage for future discussions involving black holes and their thermodynamic properties in the remaining chapters of the first part of the thesis.

To understand the low-energy theory we will consider, one must focus on massless states obtained when quantizing in a flat target space a fundamental, closed string [62, 63]. These are present both in the bosonic string and in the NS-NS sector of the superstring. They can be organized into a symmetric, traceless, rank-2 tensor, an antisymmetric rank-2 tensor, and a scalar. These modes can combine themselves into coherent states, giving rise to corresponding fields G_{MN}, B_{MN}, and Φ, which act as a background in which strings can now propagate. That background is not arbitrary, though. Consistency of the quantum theory of strings propagating in this background requires Weyl invariance to be respected, and this in turn puts strong constraints in the way the previous fields might behave: since they act as coupling constants for quantum strings, their β-functions must vanish exactly to all loops. This condition produces the background field equations, which considering only the leading order (one-loop) contribution are:

$$0 = R_{MN} + 2\nabla_M \nabla_N \Phi - \frac{1}{4} H_{MLR} H_N{}^{LR} \ , \tag{1.1a}$$

$$0 = -\frac{1}{2}\nabla^L H_{LMN} + \nabla^L \Phi H_{LMN} \ , \tag{1.1b}$$

$$0 = -\frac{1}{2}\nabla^2 \Phi + \nabla_M \Phi \nabla^M \Phi - \frac{1}{24} H_{MNL} H^{MNL} \ , \tag{1.1c}$$

where H_{MNL} is the field strength of the *Kalb-Ramond field* B_{MN}, $H_{MNL} \equiv 3\partial_{[M}B_{NL]}$. It is remarkable that the first equation is nothing but the vacuum Einstein equation modified by the presence of the extra fields. It is precisely in this way that string theory in its low-energy limit recovers classical gravity.

Although not immediately obvious, the previous set of equations has a remarkable symmetry known as T-duality when we have a background with a U(1) isometry. This transformation interchanges the fields in a very non-trivial way, and we will devote a good amount of the present chapter to it, so we postpone the presentation of the explicit transformation until the next section. Let us mention, however, that T-duality can be understood not only from the viewpoint of the low-energy effective field theory, but also as a symmetry of the string sigma model. This means that it must be respected not only by equations (1.1), but also by the field equations derived after considering higher loop corrections to the β-functions. The following question arises then: what are the possible corrections to equations (1.1) which respect T-duality? Naturally, corrections coming from some string theory (bosonic, type II or heterotic) will respect it, but in fact we will present in this chapter a more general, two-parameter family of theories which include first-order corrections to the previous equations and which present T-duality as a built-in symmetry. These were obtained in [48] – building up on earlier work [49, 50] – using the formalism of Double Field Theory (DFT), but we will recast them here in terms of the conventional low-energy fields. In this way, in the following chapters, we will have at our disposal theories which have a notion of T-duality invariance but that reduce to low-energy string theory only for some specific values of the parameters. The goal will be to use these theories to study thermodynamical quantities of black holes and their behaviour under T-duality transformations, to see whether they are preserved by it even for those values of the parameters to which we cannot attribute a string sigma model origin.

1.1 The one-loop effective action and Buscher rules

Given that equations (1.1) govern the dynamics of the target spacetime backgrounds in which strings can propagate, it is reasonable to ask whether an action exists from which these equations can be derived. The answer is in the affirmative, and the one-loop effective action has the following form:

$$\mathcal{I}_0 = \frac{1}{2\kappa_s^2}\int \mathrm{d}^D x\sqrt{-G}e^{-2\Phi}\left[R + 4\nabla^2\Phi - 4\left(\nabla\Phi\right)^2 - \frac{1}{12}H_{MNR}H^{MNR}\right] , \qquad (1.2)$$

where we included a constant κ_s, related to the D-dimensional Newton constant $8\pi G_N = 2\kappa_s^2 e^{2\Phi_0}$, Φ_0 being the constant mode of the dilaton field.[1] Extremization of this action with respect to each of the three fields G_{MN}, B_{MN}, and Φ results in the three equations (1.1).

[1]To see this, one has to conformally change the metric to go to Einstein frame, in which the dilaton does not appear as a global factor in the action: $G_{MN} \to e^{-4(\Phi-\Phi_0)/(D-2)}G_{MN}$. Notice that we work in general dimension D, although strictly speaking the target spacetime must have $D = 26$ for the bosonic string and $D = 10$ for the superstring. The general D situation might be thought to arise as a result of compactification.

As we already mentioned, the previous action possesses a symmetry whenever we have a background with a U(1) isometry: T-duality. From the viewpoint of the fundamental string, this arises because a compact, symmetric direction implies both an integral quantization of the momentum in that direction, as well as the appearance of configurations in which the string winds an integral number of times around it. The string spectrum happens to be invariant under the exchange of momentum and winding quantum numbers, provided we also change the radius of the compact direction from R_0 to α'/R_0, α' being the square of the string length. To put it another way, strings cannot distinguish between very large and very small circles. This phenomenon has a counterpart from the viewpoint of the target spacetime. We can see it following [64], which uses a rewriting of the fields adapted to the U(1) isometry. Let ψ be a coordinate adapted to the isometry, and let x^μ be coordinates in the remaining $D-1$ directions. We can write the background fields without loss of generality as:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + e^{2\sigma}\left(\mathrm{d}\psi + V_\mu\,\mathrm{d}x^\mu\right)^2 \ , \tag{1.3a}$$

$$B = \frac{1}{2}b_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + \frac{1}{2}W\wedge V + W\wedge \mathrm{d}\psi \ , \tag{1.3b}$$

$$\Phi = \phi + \frac{1}{2}\sigma \ . \tag{1.3c}$$

These expressions define the reduced fields $g_{\mu\nu}$, V_μ, $b_{\mu\nu}$, W_μ, ϕ, and σ. It is also convenient to introduced a reduced field strength $h_{\mu\nu\rho}$. This is not the field strength of $b_{\mu\nu}$, but it is the correct object to consider when looking at T-duality from these dimensionally reduced fields [64]:

$$h_{\mu\nu\rho} \equiv 3\partial_{[\mu}b_{\nu\rho]} - \frac{3}{2}W_{\mu\nu}V_\rho - \frac{3}{2}V_{\mu\nu}W_\rho \ , \tag{1.4}$$

where $V_{\mu\nu} \equiv 2\partial_{[\mu}V_{\nu]}$ and $W_{\mu\nu} \equiv 2\partial_{[\mu}W_{\nu]}$ are the usual field strengths of the vector fields appearing in the dimensional reduction. In terms of these reduced fields, the one-loop action can be shown to be given by:

$$\mathcal{I}_0 = \frac{\pi R_0}{\kappa_s^2}\int \mathrm{d}^{D-1}x\sqrt{-g}e^{-2\phi}\left[r + 4\nabla^2\phi - 4\left(\nabla\phi\right)^2 - \left(\nabla\sigma\right)^2 - \frac{1}{4}e^{2\sigma}V_{\mu\nu}V^{\mu\nu}\right.$$
$$\left. -\frac{1}{4}e^{-2\sigma}W_{\mu\nu}W^{\mu\nu} - \frac{1}{12}h_{\mu\nu\rho}h^{\mu\nu\rho}\right] \ , \tag{1.5}$$

where r is the Ricci scalar of the reduced metric, and we take the compact direction to have length $2\pi R_0$. This form of the action is manifestly invariant under the transformations:

$$\sigma \to \tilde{\sigma} = -\sigma \ , \qquad V_\mu \to \tilde{V}_\mu = W_\mu \ , \qquad W_\mu \to \tilde{W}_\mu = V_\mu \ , \tag{1.6}$$

while we keep the remaining fields invariant. These are the T-duality transformations and, being a symmetry of the action for any U(1)-invariant solution, they will also leave the equations of motion invariant. This means that these T-duality transformations will generate a new solution (the T-dual) starting from any given U(1)-symmetric background satisfying the equations of motion. In this sense, they can be thought as a solution generating technique for the low-energy equations of motion. Notice also that the transformation squares to the identity.

We can write the previous transformations in terms of the original background fields, without appealing to the dimensional reduction. In this case, we obtain the following set of rules, known as *Buscher rules* [65]:

$$\tilde{G}_{\mu\nu} = G_{\mu\nu} - \frac{G_{\psi\mu}G_{\psi\nu} - B_{\psi\mu}B_{\psi\nu}}{G_{\psi\psi}} \,, \qquad \tilde{G}_{\psi\mu} = -\frac{B_{\psi\mu}}{G_{\psi\psi}} \,, \qquad \tilde{G}_{\psi\psi} = \frac{1}{G_{\psi\psi}} \,, \tag{1.7a}$$

$$\tilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{G_{\psi\mu}B_{\psi\nu} - B_{\psi\mu}G_{\psi\nu}}{G_{\psi\psi}} \,, \qquad \tilde{B}_{\psi\mu} = -\frac{G_{\psi\mu}}{G_{\psi\psi}} \,, \tag{1.7b}$$

$$e^{-2\tilde{\Phi}} = e^{-2\Phi}G_{\psi\psi} \,. \tag{1.7c}$$

We establish the convention of denoting by tilded fields those obtained after the application of the previous rules, which are therefore the leading order T-dual fields.

Let us end this short introduction to the one-loop effective action and T-duality transformations with a brief discussion of the transformation rules for the vielbein, in case we are interested in a first order formulation of the gravitational dynamics. This might seem unnecesary at this point, but it will be crucial when discussing higher-order perturbative corrections to the action. Consider then a vielbein ${E_M}^A$ such that $G_{MN} = \eta_{AB}{E_M}^A{E_N}^B$, where we use $A, B, \ldots$ to denote flat (tangent space) indices. The dual vielbein can be shown to be:[2]

$$\tilde{E}_\mu{}^A = E_\mu{}^A - \frac{G_{\psi\mu} + B_{\psi\mu}}{G_{\psi\psi}}E_\psi{}^A \,, \qquad \tilde{E}_\psi{}^A = \frac{E_\psi{}^A}{G_{\psi\psi}} \,. \tag{1.8}$$

The same rules can be written using the dimensionally reduced fields in (1.3). In this case, we choose a reduced vielbein ${e_\mu}^a$ – where now $a, b, \ldots$ are indices in $D-1$ flat directions – such that $g_{\mu\nu} = \eta_{ab}{e_\mu}^a{e_\nu}^b$. We specify the D-dimensional vielbein to be of the form:

$$E^a = {e_\mu}^a \mathrm{d}x^\mu \,, \qquad E^\iota = e^\sigma \left(V_\mu \,\mathrm{d}x^\mu + \mathrm{d}\psi\right) \,, \tag{1.9}$$

and then the rules (1.6) apply to obtain the dual vielbein. Notice that we have introduced here a flat index ι to denote the extra direction when going to D dimensions. It is important to realize also that the rules (1.6) produce a vielbein of the same form as (1.9), and we assume this to be selected as the vilebein after the duality. This will be relevant when we compute quantities which are frame-dependent, such as the spin connection.

1.2 T-duality invariant perturbative corrections

We have shown in the previous section that the bosonic part of the leading order string effective action, (1.2), is invariant under the T-duality transformations (1.7) whenever there is an isometric direction in a given background. As we argued there, this symmetry

[2]T-duality transformations in terms of the vielbein are somewhat ill-defined, because we can pick several different equivalent frames after dualizing, provided they are related by local Lorentz transformations, which leave the spacetime metric invariant. Thus, writing a particular form of the rules involves in a sense a particular choice of dual vielbein. This might seem pedantic at this level, but it turns out to be very relevant in some situations, such as dealing with spacetime fermions [66]. In the next section we will see that this fact is also essential when including perturbative corrections to our action.

is a very fundamental one from the viewpoint of the string, and it must be respected when higher loop corrections are included in the effective action [67]. These corrections were extensively studied in the past [68, 69], and the duality invariance of the corrected actions was also discussed in several works [64, 70, 71]. Our work philosophy will be slightly different here. We will ask ourselves: starting from the fundamental bosonic background fields previously introduced, can we construct general actions which are perturbative corrections of (1.2) and which preserve invariance under T-duality?

The natural language to discuss this question is Double Field Theory (DFT). This is an idea that goes back to the works [72–74], and which naturally incorporates T-duality invariance by doubling the dimension of the space in which the fields live, in an attempt to put on an equal footing the dual character of momentum and winding modes which is at the heart of this symmetry. We do not intend to present a thorough review of DFT here,[3] but instead just to understand the basic ideas needed to build T-duality symmetric effective actions. For this purpose, we will be interested in the frame-like formulation of DFT [48, 78]. In this context, the fields of the theory are a generalized dilaton d and a generalized vielbein $\mathcal{E}_{\mathcal{M}}{}^{\mathcal{A}}$, which live in $2D$ dimensions and are parametrized as:[4]

$$e^{-2d} \equiv \sqrt{-\bar{G}}e^{-2\bar{\Phi}} \ , \tag{1.10a}$$

$$\mathcal{E}_{\mathcal{M}}{}^{\mathcal{A}} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{E}_A^{(+)M} & -G^{AB}\bar{E}_B^{(-)M} \\ \bar{E}_M^{(+)B}G_{BA} - \bar{E}_A^{(+)N}\bar{B}_{NM} & \bar{E}_M^{(-)A} + G^{AB}\bar{E}_B^{(-)N}\bar{B}_{NM} \end{pmatrix} \ . \tag{1.10b}$$

Barred fields appearing in these expressions are referred to as *DFT fields.* They are closely related to the conventional, D-dimensional ones; we will specify this relation soon. Notice also that we have two independent D-dimensional vielbeins $\bar{E}_M^{(\pm)A}$, both of which are constrained to satisfy:

$$\bar{G}_{MN} = \bar{E}_M^{(\pm)A}\bar{E}_N^{(\pm)B}G_{AB} \ , \tag{1.11}$$

with G_{AB} the D-dimensional flat metric. In DFT, the two vielbeins can be (locally) rotated independently, so there is a local symmetry group $O(1,D-1)\times O(D-1,1)$ consisting of two copies of the Lorentz group. In the previous expressions, indices $\mathcal{M},\mathcal{N},\dots$ are raised and lowered with a metric $\eta_{\mathcal{M}\mathcal{N}}$, while indices $\mathcal{A},\mathcal{B},\dots$ are raised and lowered with a metric $\eta_{\mathcal{A}\mathcal{B}}$, which have the form:

$$\eta_{\mathcal{M}\mathcal{N}} \equiv \begin{pmatrix} 0 & \delta_N^M \\ \delta_M^N & 0 \end{pmatrix} \ , \qquad \eta_{\mathcal{A}\mathcal{B}} \equiv \begin{pmatrix} G^{AB} & 0 \\ 0 & -G_{AB} \end{pmatrix} \ . \tag{1.12}$$

Furthermore, defining also the following object:

$$\mathcal{H}_{\mathcal{A}\mathcal{B}} \equiv \begin{pmatrix} G^{AB} & 0 \\ 0 & G_{AB} \end{pmatrix} \ , \tag{1.13}$$

we introduce the generalized metric $\mathcal{H}_{\mathcal{M}\mathcal{N}} = \mathcal{E}_{\mathcal{M}}{}^{\mathcal{A}}\mathcal{H}_{\mathcal{A}\mathcal{B}}\mathcal{E}_{\mathcal{N}}{}^{\mathcal{B}}$. The final objects we need to introduce are built out from the previous ones using the projectors $\mathcal{P} = (\eta+\mathcal{H})/2$ and

[3]For the interested reader, some reviews on DFT are [75–77].

[4]This particular parametrization of the $2D$-dimensional objects in terms of the D-dimensional ones is the one useful for our purposes, but other possibilities exist. Furthermore, we require everything to depend only on the D-dimensional physical coordinates, a procedure known in DFT as the *standard solution to the strong constraint.* The interested reader is encouraged to consult [48, 75] for further details.

$\bar{\mathcal{P}} = (\eta - \mathcal{H})/2$. Defining the generalized fluxes:

$$\mathcal{F}_{\mathcal{ABC}} \equiv 3\mathcal{E}_{\mathcal{M}[\mathcal{A}}\partial^{\mathcal{M}}\mathcal{E}^{\mathcal{N}}{}_{\mathcal{B}}\mathcal{E}^{\mathcal{R}}{}_{\mathcal{C}]}\eta_{\mathcal{NR}} \ , \tag{1.14}$$

we take the projections:

$$\mathcal{F}^{(-)}_{\mathcal{MAB}} \equiv \bar{\mathcal{P}}_{\mathcal{M}}{}^{\mathcal{N}}\mathcal{E}_{\mathcal{N}}{}^{\mathcal{C}}\mathcal{F}_{\mathcal{CDE}}\mathcal{P}_{\mathcal{A}}{}^{\mathcal{D}}\mathcal{P}_{\mathcal{B}}{}^{\mathcal{E}} \ , \tag{1.15a}$$

$$\mathcal{F}^{(+)}_{\mathcal{MAB}} \equiv \mathcal{P}_{\mathcal{M}}{}^{\mathcal{N}}\mathcal{E}_{\mathcal{N}}{}^{\mathcal{C}}\mathcal{F}_{\mathcal{CDE}}\bar{\mathcal{P}}_{\mathcal{A}}{}^{\mathcal{D}}\bar{\mathcal{P}}_{\mathcal{B}}{}^{\mathcal{E}} \ . \tag{1.15b}$$

There are two natural questions we must answer now: what is the relation between this formalism and T-duality transformations, and how do we reduce the DFT formulation to one which has only D-dimensional fields? Let us start with T-duality transformations. DFT naturally incorporates global $O(D,D)$ transformations $h_{\mathcal{M}}{}^{\mathcal{N}}$, defined as those preserving $\eta_{\mathcal{MN}}$:

$$\eta_{\mathcal{MN}} = h_{\mathcal{M}}{}^{\mathcal{R}}\eta_{\mathcal{RS}}h_{\mathcal{N}}{}^{\mathcal{S}} \ . \tag{1.16}$$

If we build an action out of the previously defined objects contracting all $\mathcal{M}, \mathcal{N}, \ldots$ indices with this $O(D,D)$ invariant metric, we are guaranteed that the transformation:

$$\partial_{\mathcal{M}} \to h_{\mathcal{M}}{}^{\mathcal{N}}\partial_{\mathcal{N}} \ , \qquad \mathcal{E}_{\mathcal{M}}{}^{\mathcal{A}} \to h_{\mathcal{M}}{}^{\mathcal{N}}\mathcal{E}_{\mathcal{N}}{}^{\mathcal{A}} \ , \qquad d \to d \ , \tag{1.17}$$

is going to be a symmetry. Within these $O(D,D)$ transformations, we can find the following elements:

$$h^{(k)\mathcal{N}}_{\mathcal{M}} = \begin{pmatrix} \delta^{M}{}_{N} - t^{M}{}_{N} & t^{MN} \\ t_{MN} & \delta_{M}{}^{N} - t_{M}{}^{N} \end{pmatrix} , \tag{1.18}$$

where $t = \mathrm{diag}(0,\ldots,0,1,0,\ldots,0)$ is a $D \times D$ matrix with a 1 in the k-th position. If we act with this transformation on the generalized metric:

$$\mathcal{H}_{\mathcal{MN}} = \begin{pmatrix} \bar{G}^{MN} & -\bar{G}^{MR}\bar{B}_{RN} \\ \bar{B}_{MR}\bar{G}^{RN} & \bar{G}_{MN} - \bar{B}_{MR}\bar{G}^{RS}\bar{B}_{SN} \end{pmatrix} \ , \tag{1.19}$$

and we read off the new form of the barred fields after the transformation, we find that $\bar{G}_{MN}$ and $\bar{B}_{MN}$ transform following the Buscher rules, (1.7), when the isometric coordinate ψ is the k-th one. This is also the case for the (barred) dilaton, as a consequence of the invariance of $e^{-2d} = \sqrt{-\bar{G}}e^{-2\bar{\Phi}}$. So this $O(D,D)$ transformation implements T-duality (in barred fields), and it will be a natural symmetry of any DFT action provided we contract indices using the corresponding duality invariant metric.

Let us now face the question of how we should reduce the DFT formulation to the standard D-dimensional one. In order to explain this, we should first present how the $O(1,D-1) \times O(D-1,1)$ local double Lorentz transformations act on the DFT vielbein. These (infinitesimal) transformations are parametrized by two independent Lorentz generators:

$$\Lambda_{\mathcal{A}}{}^{\mathcal{B}} = \begin{pmatrix} \Lambda^{(+)A}{}_{B} & 0 \\ 0 & \Lambda^{(-)}{}_{A}{}^{B} \end{pmatrix} \ , \tag{1.20}$$

and the DFT vielbein transforms as:

$$\delta_{\Lambda}\mathcal{E}_{\mathcal{M}}{}^{\mathcal{A}} = \mathcal{E}_{\mathcal{M}}{}^{\mathcal{B}}\Lambda_{\mathcal{B}}{}^{\mathcal{A}} \ . \tag{1.21}$$

This effectively allows for two independent local Lorentz transformations, one for each of the two vielbeins $\bar{E}_M^{(\pm)A}$. To go to the standard D-dimensional representation, we must exploit this freedom to make $\bar{E}_M^{(+)A} = \bar{E}_M^{(-)A}$, and then impose a gauge fixing condition which reduces the double Lorentz group to the single, physical one:

$$\bar{E}_M^{(+)B}\Lambda_B^{(+)A} = \bar{E}_M^{(-)B}\Lambda_B^{(-)A} = \bar{E}_M{}^B\Lambda_B{}^A \ , \tag{1.22}$$

parametrized by a single $\Lambda_B{}^A$. Notice that this step is important when discussing T-duality transformations, because (1.18) will generically produce two different vielbeins $\bar{E}_M^{(\pm)A}$ even when starting from a situation with $\bar{E}_M^{(+)A} = \bar{E}_M^{(-)A}$. The only step left to show that we recover the low energy theory of the previous section is to present an action which is invariant under the previous local double Lorentz transformations, global $O(D,D)$ transformations, and which reduces to (1.2). This action is:

$$\mathcal{I}_{\text{DFT},0} \equiv \frac{1}{2\kappa_s^2}\int \mathrm{d}X e^{-2d}\mathcal{R} \ , \tag{1.23}$$

with $\mathcal{R}$ a certain combination of fluxes and invariant metrics which is indeed invariant under the previously mentioned transformations, see [75] for details. In this case we can use the DFT parametrization (1.10) and (1.19) plus the form of $\mathcal{R}$ to show that the action is indeed the same as (1.2) when written in terms of D-dimensional fields (identifying barred and unbarred fields).

Up until now, we have made no mention to the fact that we are trying to develop a theory that accomodates perturbative corrections to the leading order effective action. As [48] shows, the key step to do this is to modify the local double Lorentz transformation (1.21) in a non-trivial way, and impose invarince of the action (much in the spirit of a gauge principle) to generate perturbative corrections. Let us then define two parameters a_- and a_+, which will be taken to be small perturbations, so we always work to linear order in them. We introduce the generalized transformation:

$$\delta_\Lambda \mathcal{E}_\mathcal{M}{}^\mathcal{A} = \mathcal{E}_\mathcal{M}{}^\mathcal{B}\Lambda_\mathcal{B}{}^\mathcal{A} + \delta'_\Lambda \mathcal{E}_\mathcal{M}{}^\mathcal{A} \ , \tag{1.24}$$

where

$$\delta'_\Lambda \mathcal{E}_\mathcal{M}{}^\mathcal{A} \equiv \left(a_- \mathcal{P}_{[\mathcal{M}}{}^\mathcal{R}\bar{\mathcal{P}}_{\mathcal{N}]}{}^\mathcal{S}\partial_\mathcal{R}\Lambda_\mathcal{C}{}^\mathcal{B}\mathcal{F}^{(-)\mathcal{C}}_{\mathcal{S}\mathcal{B}} - a_+\bar{\mathcal{P}}_{[\mathcal{M}}{}^\mathcal{R}\mathcal{P}_{\mathcal{N}]}{}^\mathcal{S}\partial_\mathcal{R}\Lambda_\mathcal{C}{}^\mathcal{B}\mathcal{F}^{(+)\mathcal{C}}_{\mathcal{S}\mathcal{B}}\right)\mathcal{E}^{\mathcal{N}\mathcal{A}} \ . \tag{1.25}$$

Notice from the parametrization (1.10) that this rule implies a very non-trivial transformation of the D-dimensional fields, which are therefore non-covariant under Lorentz transformations, even if restricted to a single copy of the group as in (1.22). This is the reason to introduce barred fields. In the presence of corrections, these DFT fields transform in a simple way under T-duality, following Buscher rules, as implied by the $O(D,D)$ transformation (1.18). But the price to pay for this simplicity is that we are dealing with fields which are not covariant under local Lorentz transformations. Only the generalized dilaton transforms as a scalar under the double local Lorentz group, so that $e^{-2d} = \sqrt{-\bar{G}}e^{-2\bar{\Phi}}$ remains invariant. Notice also that if we are dealing with a constant Lorentz transformation, $\partial_\mathcal{R}\Lambda_\mathcal{C}{}^\mathcal{B} = 0$, then $\delta'_\Lambda\mathcal{E}_\mathcal{M}{}^\mathcal{A} = 0$ and we can just consider the conventional transformation rule (1.21).

Once we have the previous transformation, [48] showed that the first order correction of the action that ensures invariance under (1.24) is:

$$\mathcal{I}_{\rm DFT} \equiv \frac{1}{2\kappa_s^2}\int \mathrm{d}X e^{-2d}\left(\mathcal{R} - 2\Lambda + a_-\mathcal{R}^{(-)} + a_+\mathcal{R}^{(+)}\right) \ . \tag{1.26}$$

Several comments are in order here. Firstly, we have included a cosmological constant term, Λ (not to be confused with the Lorentz generator, we can distinguish them from the context), based on the fact that it is invariant under $O(D,D)$ transformations on its own, and we are after the most general action satisfying this. Secondly, the perturbative corrections are weighted by the parameters $a_\pm$, and $\mathcal{R}^{(\pm)}$ are objects built out of the fluxes (1.15), the generalized metric (1.19), the generalized dilaton d, and the derivative operator. Their particular form can be found in [48] and it is not very relevant for us here, it suffices to know that they ensure (first order) invariance under anomalous local Lorentz transformations, (1.24), by cancelling the piece generated by $\mathcal{R}$. Of course, they are also constructed as $O(D,D)$ invariant contractions, thus guaranteeing T-duality invariance.

What is the form of this action in terms of D-dimensional fields? Just like before, when first order corrections were not included, we assume we have used the double local Lorentz symmetry to make the two vielbeins equal. We then reduce the group to a single copy by fixing $\Lambda^{(+)} = \Lambda^{(-)}$ and we ask how (1.26) looks like in terms of D-dimensional fields. Before showing this expression, it is useful to look at the anomalous transformation that (1.25) induces on the barred metric. Under the single copy of the Lorentz group, it can be shown to be:

$$\delta_\Lambda \bar{G}_{MN} = -\frac{1}{2}\sum_{k=\pm} a_k \Omega^{(k)}_{(MA}{}^{B}\partial_{N)}\Lambda_B{}^A \ . \tag{1.27}$$

$\Omega^{(\pm)}_{MA}{}^{B}$ are the following torsionful spin connections:

$$\Omega^{(\pm)}_{MA}{}^{B} \equiv \Omega_{MA}{}^{B} \pm \frac{1}{2}H_{MA}{}^{B} \ , \tag{1.28}$$

where $H_{MA}{}^{B} \equiv E_A{}^N H_{MN}{}^R E_R{}^B$ and $\Omega_{MA}{}^B$ is the conventional spin connection, defined as $E_A{}^M\nabla_M E_B \equiv -\Omega_{AB}{}^C E_C$, so that:

$$\Omega_{MA}{}^{B} = E_A{}^N\partial_M E_N{}^B - \Gamma^N_{MP}E_A{}^P E_N{}^B \ . \tag{1.29}$$

An important point to highlight is that we do not write bars in the object appearing in the right hand side of (1.27). This is due to the fact that the torsionful connections appear there multiplied by the perturbative parameters, so that only their leading order part is relevant to the order we are working. As we mentioned when discussing the leading order action (1.23), in that case the reduction to D dimensions identifies barred and unbarred fields, so the distinction is irrelevant in (1.27). We can now apply a clever first order field redefinition:

$$\bar{G}_{MN} = G_{MN} - \frac{1}{4}\sum_{k=\pm} a_k \Omega^{(k)}_{MA}{}^{B}\Omega^{(k)}_{NB}{}^{A} \ , \tag{1.30}$$

so that we get an unbarred metric which is invariant under local Lorentz transformations, as it should be in the D-dimensional theory. Identifying also $\bar{B}_{MN} = B_{MN}$ and

$\sqrt{-\bar{G}}e^{-2\bar{\Phi}} = \sqrt{-G}e^{-2\Phi}$, we can write the D-dimensional version of the action (1.26) in terms of unbarred fields as:

$$\mathcal{I}_{\mathrm{BdR}} = \frac{1}{2\kappa_s^2}\int \mathrm{d}^D x\sqrt{-G}e^{-2\Phi}\left[R - 2\Lambda + 4\nabla^2\Phi - 4\left(\nabla\Phi\right)^2 - \frac{1}{12}H'_{MNR}H'^{MNR} \right. \\ \left. + \frac{1}{8}\sum_{k=\pm} a_k R^{(k)}_{MNA}{}^{B} R^{(k)MN}{}_{B}{}^{A}\right] , \quad (1.31)$$

where we have defined:

$$H'_{MNR} = H_{MNR} - \frac{3}{2}\left(a_-\,\Theta^{(-)}_{MNR} - a_+\,\Theta^{(+)}_{MNR}\right) , \quad (1.32)$$

$\Theta^{(\pm)}_{MNR}$ being the gravitational Chern-Simons forms

$$\Theta^{(\pm)}_{MNR} = \Omega^{(\pm)}_{[MA}{}^{B}\partial_N\Omega^{(\pm)}_{R]B}{}^{A} + \frac{2}{3}\Omega^{(\pm)}_{[MA}{}^{B}\Omega^{(\pm)}_{NB}{}^{C}\Omega^{(\pm)}_{R]C}{}^{A} , \quad (1.33)$$

and with the Riemann tensors $R^{(\pm)}_{MNA}{}^{B}$ also built from the torsionful connections:

$$R^{(\pm)}_{MNA}{}^{B} = \partial_M\Omega^{(\pm)}_{NA}{}^{B} - \partial_N\Omega^{(\pm)}_{MA}{}^{B} + \Omega^{(\pm)}_{MA}{}^{C}\Omega^{(\pm)}_{NC}{}^{B} - \Omega^{(\pm)}_{NA}{}^{C}\Omega^{(\pm)}_{MC}{}^{B} . \quad (1.34)$$

The final form of the D-dimensional action, (1.31), is remarkable. It corresponds to a generalization of the Bergshoeff-de Roo (BdR) action [69], obtained originally for the (bosonic, non-gauge sector of the) heterotic string, corresponding to the values of the perturbative parameters $a_- = -\alpha'$, $a_+ = 0$. In fact, different values of these parameters reproduce the first order corrections of different low-energy effective actions of string theories:

$$\begin{aligned} &a_- = a_+ = -\alpha' , && \text{bosonic} , \\ &a_- = -\alpha' , \quad a_+ = 0 , && \text{heterotic} , \\ &a_- = a_+ = 0 , && \text{type II} . \end{aligned} \quad (1.35)$$

The case $a_- + a_+ = 0$ is also special, and has been studied in the literature [49]. The generalized version of the action obtained with DFT methods interpolates between these particular cases, and for generic values of the perturbative parameters a string theory sigma model origin is not known. Let us emphasize that, in (1.31), we have written things in such a way that quadratic terms in $a_\pm$ might seem to appear. This is just for convenience, we are always working to first order in the perturbative parameters and consequently quadratic terms must be discarded. The first order terms should always be thought as perturbative corrections weighted by a mass scale $M_\star$ controlling the appearence of the higher derivative corrections, so that $a_\pm \sim \mathcal{O}(M_\star^{-2})$. Naturally, in the string theory cases, $M_\star^{-2} \sim \alpha'$. Let us also make a final comment regarding local Lorentz invariance of the generalized BdR action. The redefinition (1.30) has produced a metric which is invariant under these local Lorentz transformations and, as a consequence, the associated unbarred vielbein transforms in the usual way. However, (1.24) also implies a non-trivial transformation of the $\bar{B}_{MN}$ field, and with the identification $B_{MN} = \bar{B}_{MN}$

this propagates to the unbarred field.[5] The local Lorentz symmetry of the action (1.31) takes then the following infinitesimal form:

$$\delta_\Lambda E_M{}^A = E_M{}^B \Lambda_B{}^A \ , \tag{1.36a}$$

$$\delta_\Lambda B_{MN} = -\frac{a_-}{2}\,\partial_{[M}\Lambda_A{}^B \Omega^{(-)}_{N]B}{}^A + \frac{a_+}{2}\,\partial_{[M}\Lambda_A{}^B \Omega^{(+)}_{N]B}{}^A \ . \tag{1.36b}$$

We will refer to this transformation as an *anomalous* local Lorentz transformation, due to the non-trivial B_{MN} behaviour. One can check that the previous transformation is exactly the one needed to cancel the non-trivial transformation of the gravitational Chern-Simons terms. Notice that this symmetry forces us to consider the basic objects of our theory the dilaton Φ, the B-field, and the vielbein $E_M{}^A$; since a change of vielbein must be compensanted by a corresponding transformation of the B-field.

Let us conclude this section by presenting the equations of motion derived from the action (1.31), already obtained in [1]. They can be shown to be given by:[6]

$$\begin{aligned} 0 = {} & R - 2\Lambda + 4\nabla^2\Phi - 4(\nabla\Phi)^2 \\ & - \frac{1}{12}H'_{MNR}H'^{MNR} + \frac{1}{8}\sum_{k=\pm} a_k R^{(k)}_{MNA}{}^B R^{(k)MN}{}_B{}^A \ , \end{aligned} \tag{1.37a}$$

$$\begin{aligned} 0 = {} & \nabla_M \Big[e^{-2\Phi} H'^{MNR} \\ & + \frac{3}{2}\sum_{k=\pm} a_k \left(e^{-2\Phi} H^{ST[M} R^{(k)RN]}_{ST} - k\nabla^{(k)}_S \left(e^{-2\Phi} R^{(k)S[MNR]} \right) \right) \Big] \ , \end{aligned} \tag{1.37b}$$

$$\begin{aligned} 0 = {} & R_{MN} + 2\nabla_M\nabla_N\Phi - \frac{1}{4}H'_{MRS}H'_N{}^{RS} - \frac{1}{4}\sum_{k=\pm} a_k \Big[R^{(k)}_{MRST} R^{(k)RST}_N \\ & + e^{2\Phi}\left(2G_{S(M|}\nabla_R + kH_{RS(M|}\right)\left(\delta_U{}^S \nabla^{(k)}_T + kH_{TU}{}^S\right)\left(e^{-2\Phi}R^{(k)TUR}{}_{|N)}\right)\Big] \ , \end{aligned} \tag{1.37c}$$

where $\nabla^{(k)}$ is the covariant derivative involving the connection with torsion $\Gamma^{(\pm)R}_{MN} = \Gamma^R_{MN} \mp \frac{1}{2}H_{MN}{}^R$. Notice that the last equation is obtained via variation of the action with respect to the vielbein, which is the fundamental field due to the presence of the graviatational Chern-Simons forms in the action (1.31).

1.3 Corrected T-duality transformations

The purpose of the brief incursion into the DFT formalism presented in the previous section was not only to motivate the action (1.31), but also to allow us to derive the first order corrections to the T-duality rules in terms of D-dimensional fields. This was first presented in [2], and we will reproduce the steps in this section. To set the notation from the beginning, notice that in (1.7) we used tilded fields to denote the leading order duals, obtained by means of Buscher rules. We will keep this notation here: tilded fields denote

[5] One might ask whether we can cancel the non-trivial transformation of the B-field by means of a field redefinition, just like we did for the metric. Although this is possible for the bosonic string values of the parameters, it can be shown to be impossible in general, the details can be found in [48].

[6] In order to compare with (1.1), we have combined the dilaton equation and the trace of the metric equation into a single one.

fields to which we apply Buscher rules. The full T-duality transformed fields, including the relevant first order corrections we intend to compute, will be denoted with a hat.

Let us start then with a set of basic fields which are a solution of the BdR equations of motion, (1.37). That is, we have a set $\{E_M{}^A, B_{MN}, \Phi\}$, where the metric is obtained as usual from the vielbein. To go to the DFT scheme, we only have to redefine the metric as in (1.30), and then we set $\bar{B}_{MN} = B_{MN}$, $\sqrt{-\bar{G}}e^{-2\bar{\Phi}} = \sqrt{-G}e^{-2\Phi}$. Notice that this implies we must take a barred vielbein $\bar{E}_M{}^A$ different from the original one by first order terms. The particular one we take does not matter, as long as it is a vielbein for $\bar{G}_{MN}$ which has the same leading order part as $E_M{}^A$. We denote this constraint as $(\bar{E}_M{}^A)^{(0)} = (E_M{}^A)^{(0)}$. This is due to the fact that any two vielbeins for $\bar{G}_{MN}$ differing in their first order part in $a_\pm$ must be related by a first order Lorentz transformation. These, in turn, are symmetries of the DFT theory, because the anomalous part (1.25) is already first order in the perturbative parameters, and therefore is irrelevant if the Lorentz transformation is also first order itself. To summarize, after we redefine the metric as in (1.30), we pick a vielbein for the barred metric $\bar{E}_M{}^A$ with the same leading order part as $E_M{}^A$ and any first order piece able to reproduce the first order part of $\bar{G}_{MN}$. Now we identify barred fields with those of DFT, and we set the two vielbeins to be equal, $\bar{E}_M^{(+)A} = \bar{E}_M^{(-)A} = \bar{E}_M{}^A$. Barred fields transformation under T-duality is formally identical to Buscher rules, (1.7), since the $O(D,D)$ transformation that generates it is the same, (1.18). The only subtlety comes from the fact that the two vielbeins have different transformations, [66]:

$$\hat{\bar{E}}_\mu^{(\pm)A} = \bar{E}_\mu{}^A - \frac{\bar{Q}^{(\mp)}_{\psi\mu}}{\bar{G}_{\psi\psi}}\bar{E}_\psi{}^A \ , \qquad \hat{\bar{E}}_\psi^{(\pm)A} = \mp\frac{\bar{E}_\psi{}^A}{\bar{G}_{\psi\psi}} \ , \tag{1.38}$$

where

$$\bar{Q}^{(\pm)}_{\psi\mu} \equiv \bar{G}_{\psi\mu} \pm \bar{B}_{\psi\mu} \ . \tag{1.39}$$

Albeit not obvious at first glance, notice that both dual vielbeins lead to the same metric. When discussing leading order transformations of the vielbein in (1.8), we chose to rotate the plus vielbein to make it coincide with the minus one. We must do the same now in order to reduce the theory to D dimensions from DFT, so we look for a Lorentz transformation that makes $\hat{\bar{E}}_M^{(+)A}$ and $\hat{\bar{E}}_M^{(-)A}$ equal. The transformation we are looking for is actually an element of the double Lorentz group of DFT (non-trivial only in one of the copies, the one which affects $\hat{\bar{E}}_M^{(+)A}$), so it induces a change of the form (1.24), including the anomalous first order part. Were this not the case, we could easily find a conventional finite Lorentz transformation relating the two vielbeins:

$$\hat{\bar{E}}_M^{(-)A} = \hat{\bar{E}}_M^{(+)B}\bar{\mathcal{L}}_B{}^A \ , \qquad \bar{\mathcal{L}}_B{}^A \equiv \delta_B{}^A - 2\frac{\bar{E}_{\psi B}\bar{E}_\psi{}^A}{\bar{G}_{\psi\psi}} \ , \tag{1.40}$$

which satisfies $\bar{\mathcal{L}}_B{}^C\bar{\mathcal{L}}_C{}^A = \delta_B{}^A$ and $\det \bar{\mathcal{L}}_B{}^A = -1$, [66]. But we seem to be in trouble here: (1.24) is the real transformation we must perform to make the vielbeins equal, and not a conventional Lorentz transformation. Furthermore, (1.24) is only an infinitesimal version of the anomalous Lorentz transformation, and it cannot be easily exponentiated. But we will need a finite transformation to equate the vielbeins. This problem can be overcome, though, for a vielbein of the form (1.9), which has the nice property that $\bar{E}_M{}^A$

can also be taken with the same form, thereby $\bar{\mathcal{L}}_B{}^A = \mathrm{diag}(1,\ldots,1,-1)$. As mentioned earlier, these uniform Lorentz transformations, $\partial_M \bar{\mathcal{L}}_B{}^A = 0$, are symmetries of the full action in DFT and entail no anomalous modification of the fields. Consequently, we can safely choose $\hat{\bar{E}}_M{}^A = \hat{\bar{E}}_M^{(-)A}$ as the dual vielbein in the D-dimensional theory written in barred fields. Notice that due to our particular choice of vielbein to make (1.40) a valid symmetry transformation in the presence of corrections, we will not derive the T-duality rules for a generic D-dimensional vielbein. It must be taken as having the form adapted to the dimensional reduction, (1.9).

Knowing that no anomalous Lorentz transformation appears in the process of equating the two DFT vielbeins, and that the barred vielbein transformation is the same as the leading order one, we can write the rules to dualize $\bar{G}_{MN}$, $\bar{B}_{MN}$, and $\bar{\Phi}_{MN}$. They will be formally equal to the Buscher rules:

$$\hat{\bar{G}}_{MN} = \tilde{\bar{G}}_{MN} \ , \qquad \hat{\bar{B}}_{MN} = \tilde{\bar{B}}_{MN} \ , \qquad e^{-2\hat{\bar{\Phi}}}\sqrt{-\hat{\bar{G}}} = e^{-2\tilde{\bar{\Phi}}}\sqrt{-\tilde{\bar{G}}} \ . \tag{1.41}$$

At this point, it only remains to relate the dual barred fields to the unbarred ones. Let us illustrate this with full level of detail for one of the components of the metric, since the remaining ones will be identical. Using Buscher rules, the previous equation gives us:

$$\hat{\bar{G}}_{\psi\psi} = \frac{1}{\bar{G}_{\psi\psi}} \ . \tag{1.42}$$

We can now use the relation between barred and unbarred fields, which only affects the metric as defined in (1.30). Working to first order in the couplings:

$$\hat{G}_{\psi\psi} - \frac{1}{4}\sum_{k=\pm} a_k \hat{\Omega}^{(k)2}_{\psi\psi} = \frac{1}{G_{\psi\psi}} + \frac{1}{4}\sum_{k=\pm} a_k \frac{\Omega^{(k)2}_{\psi\psi}}{G^2_{\psi\psi}} \ , \tag{1.43}$$

where

$$\Omega^{(k)2}_{MN} \equiv \Omega^{(k)}_{MA}{}^B \Omega^{(k)}_{NM}{}^A \ , \tag{1.44}$$

and a similar definition holds for the hatted field. We can then write $\tilde{\Omega}^{(k)2}_{\psi\psi}$ instead of $\hat{\Omega}^{(k)2}_{\psi\psi}$ in the left hand side of the previous equation, since the difference between those two objects is first order in the couplings and $\hat{\Omega}^{(k)2}_{\psi\psi}$ appears multiplied by a_k. Thus, we arrive at the final expression:

$$\hat{G}_{\psi\psi} = \frac{1}{G_{\psi\psi}} + \sum_{k=\pm} \frac{a_k}{4}\left(\tilde{\Omega}^{(k)2}_{\psi\psi} + \frac{\Omega^{(k)2}_{\psi\psi}}{G^2_{\psi\psi}}\right) \ , \tag{1.45}$$

where $\tilde{\Omega}^{(k)2}_{\psi\psi}$ has to be obtained from the connection associated with the Buscher transformed vielbein, (1.8). Some useful results to do this transformation directly on the connection can be obtained by looking at $\Omega_{ABC} = E_A{}^M \Omega_{MBC}$ in terms of the fields of the dimensional reduction:

$$\Omega_{abc} = \omega_{abc} \ , \quad \Omega_{ab\iota} = \frac{e^{\sigma} V_{ab}}{2} = -\Omega_{\iota ab} \ , \quad \Omega_{\iota a\iota} = -\partial_a \sigma \ , \tag{1.46}$$

where $\omega_{\mu a}{}^{b}$ is the connection of $e_{\mu}{}^{a}$, and all lower case flat indices are related to curved ones using $e_{\mu}{}^{a}$. From these expressions we obtain:

$$\Omega^{(\pm)}_{abc} = \omega_{abc} \pm \frac{1}{2} h_{abc} \ , \qquad \Omega^{(\pm)}_{\iota a \iota} = -\partial_a \sigma \ , \tag{1.47a}$$

$$\Omega^{(\pm)}_{ab\iota} = \frac{1}{2}\left(e^{\sigma} V_{ab} \pm e^{-\sigma} W_{ab}\right) = -\Omega^{(\mp)}_{\iota ab} \ . \tag{1.47b}$$

Buscher rules in terms of dimensionally reduced fields, (1.6), produce then the following simple form of the leading order dual connection:

$$\tilde{\Omega}^{(-)}_{abc} = \Omega^{(-)}_{abc} \ , \qquad \tilde{\Omega}^{(-)}_{\iota ab} = \Omega^{(-)}_{\iota ab} \ , \qquad \tilde{\Omega}^{(-)}_{ab\iota} = -\Omega^{(-)}_{ab\iota} \ , \qquad \tilde{\Omega}^{(-)}_{\iota a\iota} = -\Omega^{(-)}_{\iota a\iota} \ , \tag{1.48a}$$

$$\tilde{\Omega}^{(+)}_{abc} = \Omega^{(+)}_{abc} \ , \qquad \tilde{\Omega}^{(+)}_{\iota ab} = -\Omega^{(+)}_{\iota ab} \ , \qquad \tilde{\Omega}^{(+)}_{ab\iota} = \Omega^{(+)}_{ab\iota} \ , \qquad \tilde{\Omega}^{(+)}_{\iota a\iota} = -\Omega^{(+)}_{\iota a\iota} \ . \tag{1.48b}$$

With these rules, it is immediate to obtain $\tilde{\Omega}^{(k)2}_{MN} = E_M{}^A E_N{}^B \tilde{\Omega}^{(k)2}_{AB}$.

Much in the same way we obtained (1.45), we can proceed to derive the remaining transformation rules. We collect here the results, writing some of the terms as leading order Buscher rules, (1.7), so that the final expressions are more compact:

$$\hat{G}_{\mu\nu} = \tilde{G}_{\mu\nu} + \sum_{k=\pm} \frac{a_k}{4} \left(\tilde{\Omega}^{(k)2}_{\mu\nu} - \Omega^{(k)2}_{\mu\nu} + \frac{2\Omega^{(k)2}_{\psi(\mu} G_{\nu)\psi}}{G_{\psi\psi}} - \frac{\Omega^{(k)2}_{\psi\psi}}{G^2_{\psi\psi}} \left(G_{\psi\mu} G_{\psi\nu} - B_{\psi\mu} B_{\psi\nu}\right) \right) \ , \tag{1.49a}$$

$$\hat{G}_{\psi\mu} = \tilde{G}_{\psi\mu} + \sum_{k=\pm} \frac{a_k}{4} \left(\tilde{\Omega}^{(k)2}_{\psi\mu} - \frac{\Omega^{(k)2}_{\psi\psi} B_{\psi\mu}}{G^2_{\psi\psi}} \right) \ , \tag{1.49b}$$

$$\hat{G}_{\psi\psi} = \tilde{G}_{\psi\psi} + \sum_{k=\pm} \frac{a_k}{4} \left(\tilde{\Omega}^{(k)2}_{\psi\psi} + \frac{\Omega^{(k)2}_{\psi\psi}}{G^2_{\psi\psi}} \right) \ , \tag{1.49c}$$

$$\hat{B}_{\mu\nu} = \tilde{B}_{\mu\nu} + \sum_{k=\pm} \frac{a_k}{4} \frac{2}{G_{\psi\psi}} \left(\Omega^{(k)2}_{\psi[\mu} - \frac{\Omega^{(k)2}_{\psi\psi}}{G_{\psi\psi}} G_{\psi[\mu} \right) B_{\psi\nu]} \ , \tag{1.49d}$$

$$\hat{B}_{\psi\mu} = \tilde{B}_{\psi\mu} + \sum_{k=\pm} \frac{a_k}{4} \frac{1}{G_{\psi\psi}} \left(\tilde{\Omega}^{(k)2}_{\psi\mu} - \frac{\Omega^{(k)2}_{\psi\psi} G_{\psi\mu}}{G_{\psi\psi}} \right) \ , \tag{1.49e}$$

$$e^{-2\hat{\Phi}} \sqrt{-\hat{G}} = e^{-2\Phi} \sqrt{-G} \ . \tag{1.49f}$$

This last transformation rule is the only one obtained through a different procedure to the one used before to derive $\hat{G}_{\psi\psi}$. It comes from the following chain of identities, derived from the fact that the DFT dilaton d is preserved both in the transformation between barred and unbarred fields and in the $O(D, D)$ T-duality transformation:

$$e^{-2\hat{\Phi}} \sqrt{-\hat{G}} = e^{-2\hat{\bar{\Phi}}} \sqrt{-\hat{\bar{G}}} = e^{-2\bar{\Phi}} \sqrt{-\bar{G}} = e^{-2\Phi} \sqrt{-G} \ . \tag{1.50}$$

Let us emphasize once again that the metric is not one of the fundamental objects of the D-dimensional theory (1.31): we need to specify a vielbein. Therefore, the previous set of rules is not complete, and we must determine how we choose $\hat{E}_M{}^A$. In barred fields we already chose $\hat{\bar{E}}_M{}^A = \hat{\bar{E}}_M^{(-)A}$ as the dual to reduce to D-dimensions from DFT, so:

$$\hat{\bar{E}}_\mu{}^A = \bar{E}_\mu{}^A - \frac{\bar{Q}^{(+)}_{\psi\mu}}{\bar{G}_{\psi\psi}} \bar{E}_\psi{}^A \,, \qquad \hat{\bar{E}}_\psi{}^A = \frac{\bar{E}_\psi{}^A}{\bar{G}_{\psi\psi}} \,. \tag{1.51}$$

We also mentioned that $\bar{E}_M{}^A$ must be chosen with the leading order part determined by the unbarred vielbein, $(\bar{E}_M{}^A)^{(0)} = (E_M{}^A)^{(0)}$, while the first order part is arbitrary provided it reproduces $\bar{G}_{MN}$. The same thing happens when going from $\hat{\bar{E}}_M{}^A$ to $\hat{E}_M{}^A$: the transformation (1.30), applied to go from $\hat{\bar{G}}_{MN}$ to $\hat{G}_{MN}$, forces a first order change of the metric, while it assumes the leading order part of the vielbein to remain unchanged. Thus, we must choose a vielbein $\hat{E}_M{}^A$ that reproduces the leading order part of $\hat{\bar{E}}_M{}^A$ while at the same time being a vielbein of the hatted metric. The particular first order part of $\hat{E}_M{}^A$ we choose is irrelevant provided it satisfies the previous constraint: vielbeins for $\hat{G}_{MN}$ differing in their first order part will be related by first order Lorentz transformations, and those are symmetries of our theory, (1.31). In summary, tracing out the full transformation of the leading order part of the vielbein, it must be the same as the one given by Buscher rules, (1.8):

$$(\hat{E}_\mu{}^A)^{(0)} = E_\mu{}^A - \frac{Q^{(+)}_{\psi\mu}}{G_{\psi\psi}} E_\psi{}^A \,, \qquad (\hat{E}_\psi{}^A)^{(0)} = \frac{E_\psi{}^A}{G_{\psi\psi}} \,, \tag{1.52}$$

while the first order part, $(\hat{E}_M{}^A)^{(1)}$, can be chosen at will provided the full vielbein is a valid vielbein of $\hat{G}_{MN}$ as given by the rules (1.49).

1.4 Field redefinitions and corrected T-duality rules

When going from the DFT formulation of our theory to a traditional D-dimensional scheme, it was important to redefine our metric as in (1.30) to get a field invariant under local Lorentz transformations. Although it would have been odd, in principle we could have stuck with the barred metric field: once we make the two vielbeins of the DFT formulation equal, nothing prevents us from reducing to a D-dimensional theory. The action obtained would not have been (1.31), and the metric would not have been Lorentz invariant, but it could in principle be done. And given that barred fields transform according to Buscher rules, and that the compensating Lorentz transformation to make the vielbeins equal (1.40) has no anomalous part if we choose a vielbein of the form (1.9), barred fields in the presence of corrections would transform simply following the Buscher rules, provided the restriction on the form of the vielbein is respected. We pay the price of a Lorentz non-covariant metric, but we get a simpler set of transformations under T-duality.

The underlying mechanism of redefining the fields is allowed more generally, and different redefinitions produce fields with properties which are better suited for different problems or discussions. It is conventional to leave the leading order part of the action, (1.2), invariant.[7] Therefore, we will only consider field redefinitions which are first order

[7] An exception is the redefinition that takes us from the string frame to the Einstein frame, already mentioned and which has the form $G_{MN} \to e^{-4(\Phi-\Phi_0)/(D-2)} G_{MN}$.

in the parameters $a_\pm$. Dimensional analysis and the symmetries of the different fields make the following the most general field redefinition allowed:

$$\begin{aligned}
G^{(\mathrm{S})}_{MN} &= G^{(\mathrm{BdR})}_{MN} - a_1 R_{MN} - a_2 H_{MRS} H_N{}^{RS} - a_3 \nabla_M \Phi \nabla_N \Phi - a_4 \nabla_M \nabla_N \Phi \\
&\qquad - a_5 \Omega_{MA}{}^{B} \Omega_{NB}{}^{A} - a_6 \Omega_{(M}{}^{AB} H_{N)AB} \ , \qquad (1.53a)\\
B^{(\mathrm{S})}_{MN} &= B^{(\mathrm{BdR})}_{MN} - b_1 \nabla^R H_{MNR} - b_2 H_{MNR} \nabla^R \Phi - b_3 \Omega_{[M}{}^{AB} H_{N]AB} \ , \qquad (1.53b)\\
\Phi^{(\mathrm{S})} &= \Phi^{(\mathrm{BdR})} - c_1 R - c_2 H_{MNR} H^{MNR} - c_3 (\nabla\Phi)^2 - c_4 \nabla^2 \Phi \ . \qquad (1.53c)
\end{aligned}$$

As already mentioned, we are considering the parameters of this field redefinitions to be of the same order as the couplings $a_\pm$, that is, $\mathcal{O}(M_\star^{-2})$. In this section, we also include a superscript with parentheses identifying the scheme in which we are working, meaning the particular fields we are considering. BdR means the unbarred fields of previous sections, which satisfy the equations of motion (1.37) and which transform under T-duality as in (1.49). S will mean any other scheme, defined by field redefinitions of the previous form. Notice that we do not need to specify the scheme in which the fields multiplied by the parameters in the previous redefinitions appear. This is because we only consider first order field redefinitions and therefore the redefinition terms are only sensitive to the leading order part of the fields, which is kept fixed independently of the scheme.

T-duality rules for the BdR scheme with a vielbein of the form (1.9) were presented in (1.49).[8] In [3], we obtained the T-duality rules in a general scheme S, related to the BdR one as in (1.53). The results are:

$$\begin{aligned}
\hat\sigma^{(\mathrm{S})} =& -\sigma^{(\mathrm{S})} + \left(a_1 - \frac{a_4}{2} + 2a_5 + 2\gamma_+\right)(\mathcal{D}\sigma)^2 - \frac{1}{8}(a_1 + 4a_2 - a_5 - 2\gamma_+) \times \qquad (1.54a)\\
&\times \left(e^{2\sigma} V^{\lambda\rho} V_{\lambda\rho} + e^{-2\sigma} W^{\lambda\rho} W_{\lambda\rho}\right) - \frac{1}{2}(\gamma_- - a_6) V^{\lambda\rho} W_{\lambda\rho} \ ,\\
\hat V^{(\mathrm{S})}_\mu =& W^{(\mathrm{S})}_\mu + \frac{1}{2}(\gamma_+ - b_3 + a_5) W_b{}^a \omega_{\mu a}{}^b + \frac{e^{2\sigma}}{4}(-4a_2 + 2b_1 + b_3 + \gamma_+) h_{\mu\lambda\rho} V^{\lambda\rho} \qquad (1.54b)\\
&+ \frac{1}{4}(6a_1 - a_4 + 4a_5 + 4b_1 - 2b_2 + 4b_3 + 4\gamma_+) W_{\mu\rho} \mathcal{D}^\rho\sigma + \frac{1}{2}(a_4 - 2b_2) W_{\mu\rho} \mathcal{D}^\rho \phi\\
&- \frac{1}{2}(a_1 + 2b_1) \mathcal{D}^\rho W_{\mu\rho} - \frac{1}{2}(\gamma_- - a_6)\left(e^{2\sigma} V_b{}^a \omega_{\mu a}{}^b + \frac{1}{2} h_{\mu\lambda\rho} W^{\lambda\rho} - 2e^{2\sigma} V_{\mu\rho} \mathcal{D}^\rho \sigma\right) \ ,\\
\hat W^{(\mathrm{S})}_\mu =& V^{(\mathrm{S})}_\mu - \frac{1}{2}(\gamma_+ - b_3 + a_5) V_b{}^a \omega_{\mu a}{}^b - \frac{e^{-2\sigma}}{4}(-4a_2 + 2b_1 + b_3 + \gamma_+) h_{\mu\lambda\rho} W^{\lambda\rho} \qquad (1.54c)\\
&+ \frac{1}{4}(6a_1 - a_4 + 4a_5 + 4b_1 - 2b_2 + 4b_3 + 4\gamma_+) V_{\mu\rho} \mathcal{D}^\rho\sigma - \frac{1}{2}(a_4 - 2b_2) V_{\mu\rho} \mathcal{D}^\rho \phi\\
&+ \frac{1}{2}(a_1 + 2b_1) \mathcal{D}^\rho V_{\mu\rho} + \frac{1}{2}(\gamma_- - a_6)\left(e^{-2\sigma} W_b{}^a \omega_{\mu a}{}^b + \frac{1}{2} h_{\mu\lambda\rho} V^{\lambda\rho} + 2e^{-2\sigma} W_{\mu\rho} \mathcal{D}^\rho \sigma\right) \ ,\\
\hat\phi^{(\mathrm{S})} =& \phi^{(\mathrm{S})} - \frac{1}{16}(a_1 - 4a_2 - a_5 + 4c_1 + 48c_2)\left(e^{2\sigma} V_{\lambda\rho} V^{\lambda\rho} - e^{-2\sigma} W_{\lambda\rho} W^{\lambda\rho}\right) \qquad (1.54d)\\
&+ \frac{1}{2}(a_1 - 8c_1 + 2c_4) \mathcal{D}^2\sigma - \frac{1}{2}(a_4 - 4c_3 - 4c_4) \mathcal{D}_\rho \sigma \mathcal{D}^\rho \phi \ ,\\
\hat g^{(\mathrm{S})}_{\mu\nu} =& g^{(\mathrm{S})}_{\mu\nu} - \frac{1}{2}(a_1 + 4a_2 + a_5)\left(e^{2\sigma} V_{\mu\rho} V_\nu{}^\rho - e^{-2\sigma} W_{\mu\rho} W_\nu{}^\rho\right) \qquad (1.54e)
\end{aligned}$$

[8] Although not written in terms of the reduced fields, it is clear that the transformation of σ can be read off from $\hat G_{\psi\psi}$, that of V_μ from $\hat G_{\psi\mu}$, and that of $g_{\mu\nu}$ from $\hat G_{\mu\nu}$. Similarly, we obtain the transformations of W_μ, $b_{\mu\nu}$, and ϕ from $\hat B_{\psi\mu}$, $\hat B_{\mu\nu}$, and $\hat\Phi$, respectively.

$$
\begin{aligned}
&+ (-2a_1 + a_4)\, \mathcal{D}_\mu \mathcal{D}_\nu \sigma + 2a_3 \mathcal{D}_{(\mu} \sigma \mathcal{D}_{\nu)} \phi \, , \\
\hat{b}^{(\mathrm{S})}_{\mu\nu} = b^{(\mathrm{S})}_{\mu\nu} &- \frac{1}{2} (\gamma_+ - b_3 + a_5) \left(V_b{}^a \omega_{[\mu a}{}^b W_{\nu]} - W_b{}^a \omega_{[\mu\alpha}{}^b V_{\nu]} \right) \qquad (1.54\mathrm{f}) \\
&+ \frac{1}{2} (a_1 + 2b_1) \left(\mathcal{D}^\rho W_{\rho[\mu} V_{\nu]} - \mathcal{D}^\rho V_{\rho[\mu} W_{\nu]} \right) + (2b_1 + b_2)\, h_{\mu\nu\rho} \mathcal{D}^\rho \sigma \\
&+ \frac{1}{4} (4a_2 - 2b_1 - b_3 - \gamma_+) \left(e^{2\sigma} V_{[\mu} h_{\nu]\lambda\rho} V^{\lambda\rho} - e^{-2\sigma} W_{[\mu} h_{\nu]\lambda\rho} W^{\lambda\rho} \right) \\
&+ \frac{1}{4} (-6a_1 + a_4 - 4a_5 - 4b_1 + 2b_2 - 4b_3 - 4\gamma_+) \left(V_{[\mu} W_{\nu]\rho} \mathcal{D}^\rho \sigma + W_{[\mu} V_{\nu]\rho} \mathcal{D}^\rho \sigma \right) \\
&- \frac{1}{2} (a_4 - 2b_2) \left(V_{[\mu} W_{\nu]\rho} \mathcal{D}^\rho \phi - W_{[\mu} V_{\nu]\rho} \mathcal{D}^\rho \phi \right) - 2b_3 V^\rho_{\ [\mu} W_{\nu]\rho} \\
&+ \frac{1}{2} (\gamma_- - a_6) \Bigg(e^{-2\sigma} W_b{}^a \omega_{[\mu a}{}^b W_{\nu]} - e^{2\sigma} V_b{}^a \omega_{[\mu a}{}^b V_{\nu]} - \frac{1}{2} W^{\lambda\rho} h_{\lambda\rho[\mu} V_{\nu]} \\
&\qquad + \frac{1}{2} V^{\lambda\rho} h_{\lambda\rho[\mu} W_{\nu]} - 2e^{-2\sigma} W_{[\mu} W_{\nu]\rho} \mathcal{D}^\rho \sigma - 2e^{2\sigma} V_{[\mu} V_{\nu]\rho} \mathcal{D}^\rho \sigma \Bigg) \, .
\end{aligned}
$$

In these expressions, $\mathcal{D}$ denotes the covariant derivative with respect to the $(D-1)$-dimensional metric $g_{\mu\nu}$, and $\omega_{\mu a}{}^b$ is the spin connection associated with the reduced vielbein $e_\mu{}^a$. We have also introduced new parameters $\gamma_\pm$ linearly related to the previous ones, $a_\pm$, by:

$$
\gamma_\pm \equiv \mp \frac{a_- \pm a_+}{4} \, . \qquad (1.55)
$$

Writing the previous rules for generic $a_1, \ldots, c_4$ coefficients, or in other words translating them into new schemes starting from a given one, is a straightforward but tedious task. We will illustrate the procedure with one of the fields, say σ. The first step consists of computing all tensors in the dimensional reduction defined by (1.3) and (1.9). After that is done, we can start from a scheme where the rules are known, say the BdR one, where $\hat{\sigma}^{(\mathrm{BdR})} = -\sigma^{(\mathrm{BdR})} + M_\star^{-2}\xi$ for a certain ξ. To obtain the rules in scheme S related as $\sigma^{(\mathrm{S})} = \sigma^{(\mathrm{BdR})} + M_\star^{-2}s$ for some s, we just have to compute $\hat{\sigma}^{(\mathrm{S})} = \hat{\sigma}^{(\mathrm{BdR})} + M_\star^{-2}\hat{s} = -\sigma^{(\mathrm{BdR})} + M_\star^{-2}(\xi + \tilde{s}) = -\sigma^{(\mathrm{S})} + M_\star^{-2}(\xi + \tilde{s} + s)$. Notice that the fields themselves may have some explicit, first order in $M_\star^{-2}$ dependence.

We will conclude this section with a brief discussion of some interesting schemes, most of them commonly discussed in the literature. First of all, notice that it is possible to tune the coefficients in order to set to zero all corrections to the T-duality transformations. For generic $\gamma_\pm$ it is enough to set:

$$
\begin{aligned}
&a_2 = -\frac{a_1}{4} + \frac{\gamma_+}{4} \, , \quad a_4 = 2a_1 \, , \quad a_5 = -\gamma_+ \, , \qquad a_6 = \gamma_- \, , \qquad a_3 = b_3 = 0 \, , \\
&b_1 = -\frac{a_1}{2} \, , \qquad b_2 = a_1 \, , \quad c_2 = -\frac{a_1}{24} - \frac{c_1}{12} \, , \quad c_3 = a_1 - 4c_1 \, , \quad c_4 = 4c_1 - \frac{a_1}{2} \, ,
\end{aligned}
$$

and T-duality transformations reduce to Buscher rules even with the corrections. This is expected based on the DFT formulation, as discussed at the beginning of the present section. In fact, setting $a_1 = c_1 = 0$, the field redefinition is given by $a_2 = \gamma_+/4$, $a_5 = -\gamma_+$, and $a_6 = \gamma_-$. In this case, the redefinition (1.53) is just the same as (1.30), so the scheme is actually the barred DFT scheme.[9] As already mentioned, T-duality is quite

[9] Recall that the DFT scheme allows for any choice of vielbein for the barred metric, while in this

simple to perform in this scheme, but the fact that the spin connection appears in the field redefinition of the metric makes it a non-covariant scheme under Lorentz transformations.

Another interesting scheme is that of Metsaev and Tseytlin [68], generalized in [48] to generic values of the $a_\pm$ (or $\gamma_\pm$) parameters. The field redefinition is:

$$G^{(\mathrm{MT})}_{MN} = G^{(\mathrm{BdR})}_{MN} + \frac{\gamma_+}{2} H_{MRS} H_N{}^{RS} \ , \tag{1.56a}$$

$$\begin{aligned} B^{(\mathrm{MT})}_{MN} &= B^{(\mathrm{BdR})}_{MN} + \gamma_+ \left(\nabla^R H_{MNR} - 2 H_{MNR} \nabla^R \Phi - \Omega_{[M}{}^{AB} H_{N]AB} \right) \\ &\cong B^{(\mathrm{BdR})}_{MN} - \gamma_+ \Omega_{[M}{}^{AB} H_{N]AB} \ , \end{aligned} \tag{1.56b}$$

$$\Phi^{(\mathrm{MT})} = \Phi^{(\mathrm{BdR})} + \frac{\gamma_+}{8} H_{MNR} H^{MNR} \ , \tag{1.56c}$$

where in the B-field redefinition $\cong$ means that the equations of motion were used to simplify the term in parentheses (being it multiplied by γ_+, it is enough to use the one-loop equations of motion, (1.1)). Since T-duality rules are to be applied on-shell, we can safely use the equations of motion in the redefinition if our only goal is to read those rules. They are then given by (1.54) with $a_2 = -\gamma_+/2$, $b_3 = \gamma_+$, $c_2 = -\gamma_+/8$, and the remaining coefficients equal to zero. The interest in this generalized Metsaev-Tseytlin scheme comes from the fact that the action takes a fairly simple form:

$$\mathcal{I}_{\mathrm{MT}} = \frac{1}{2\kappa_s^2} \int \mathrm{d}^D x \sqrt{-G} e^{-2\Phi} \left(L^0_{\mathrm{MT}} + \gamma_+ L^{(+)}_{\mathrm{MT}} + \gamma_- L^{(-)}_{\mathrm{MT}} \right) \ , \tag{1.57}$$

where

$$L^0_{\mathrm{MT}} = R - 2\Lambda + 4\nabla^2\Phi - 4(\nabla\Phi)^2 - \frac{1}{12} H_{MNR} H^{MNR}, \tag{1.58a}$$

$$\begin{aligned} L^{(+)}_{\mathrm{MT}} = \frac{1}{2} \Bigg(& R_{MNRS} R^{MNRS} - \frac{1}{2} H^{MNR} H_{MSL} R_{NR}{}^{SL} \\ & + \frac{1}{24} H^{MNR} H_{MS}{}^T H_{NT}{}^W H_{RW}{}^S - \frac{1}{8} H_M{}^{RS} H_{NRS} H^{MTW} H^N{}_{TW} \Bigg) \ , \end{aligned} \tag{1.58b}$$

$$L^{(-)}_{\mathrm{MT}} = H_{MNR} \Theta^{MNR} \ , \tag{1.58c}$$

all the fields in the previous equations being MT fields. We refer the reader to [48] for further details.

Finally, the rules above can be compared to the ones first derived by Kaloper and Meissner in [64] for the bosonic string ($\gamma_+ = \alpha'/2$, $\gamma_- = 0$). The scheme used is obtained setting the coefficients to:

$$a_1 = \alpha' \ , \quad a_2 = -\frac{\alpha'}{4} \ , \quad b_1 = -b_3 = -\frac{\alpha'}{2} \ , \quad c_1 = \frac{\alpha'}{8} \ , \quad c_2 = -\frac{5\alpha'}{96} \ , \quad c_3 = -\frac{\alpha'}{2} \ ,$$

and the rest of them equal to zero. To match results, one has to take into account the possibility of transforming the reduced fields by doing diffeomorphisms and gauge

section and the previous one we assume we are in a gauge-fixed scheme, in which the vielbein must be of the form (1.9). This is so in order to obtain the rules (1.49), and also to be able to write them in terms of dimensionally reduced fields.

transformations. Under such symmetries, the T-dual reduced fields transform as:

$$\hat{V} \to \hat{V} + \alpha' \left(\mathcal{L}_\xi W + \mathrm{d}v\right) \ , \tag{1.59}$$

$$\hat{W} \to \hat{W} + \alpha' \left(\mathcal{L}_\xi V + \mathrm{d}w\right) \ , \tag{1.60}$$

$$\hat{b} \to \hat{b} + \alpha' \left(\mathcal{L}_\xi b + \mathrm{d}\beta + \frac{1}{2} V \wedge \mathrm{d}v + \frac{1}{2} W \wedge \mathrm{d}w\right) \ , \tag{1.61}$$

and the remaining fields transform normally under diffemorphisms. We are restricting to transformations which are first order in α', both for diffeomorphims and gauge transformations. The $\mathrm{d}w$ and $\mathrm{d}\beta$ terms come from gauge transformations of the B-field with parameter $\beta_\mu \mathrm{d}x^\mu + w\, \mathrm{d}\psi$, while v appears when including diffeomorphisms of the form $\psi \to \psi + \alpha' v$. Choosing the following set of parameters:

$$\xi^\mu = \mathcal{D}^\mu \sigma \ , \quad w = -V_\nu\, \mathcal{D}^\nu \sigma \ , \quad v = -W_\nu\, \mathcal{D}^\nu \sigma \ , \quad \beta_\mu = \left(b_{\mu\nu} - V_{(\mu}\, W_{\nu)}\right) \mathcal{D}^\nu \sigma \ , \tag{1.62}$$

we obtain the following set of rules in the KM scheme:

$$\hat{\sigma} = -\,\sigma + \frac{\alpha'}{2}\left[\frac{e^{2\sigma}}{4} V_{\lambda\rho}\, V^{\lambda\rho} + \frac{e^{-2\sigma}}{4} W_{\lambda\rho}\, W^{\lambda\rho} + 2\left(\mathcal{D}\sigma\right)^2\right] \ , \tag{1.63}$$

$$\hat{V}_\mu = W_\mu + \frac{\alpha'}{2}\left[\frac{e^{2\sigma}}{2} h_{\mu\lambda\rho} V^{\lambda\rho} + 2 W_{\mu\rho}\, \mathcal{D}^\rho \sigma\right] \ , \tag{1.64}$$

$$\hat{W}_\mu = V_\mu - \frac{\alpha'}{2}\left[\frac{e^{-2\sigma}}{2} h_{\mu\lambda\rho} W^{\lambda\rho} - 2 V_{\mu\rho}\, \mathcal{D}^\rho \sigma\right] , \tag{1.65}$$

$$\begin{aligned}\hat{b}_{\mu\nu} = b_{\mu\nu} + \alpha' \Big[& V_{[\mu}{}^{\rho} W_{\nu]\rho} - \left(V_{[\mu}\, W_{\nu]\rho} + W_{[\mu}\, V_{\nu]\rho}\right) \mathcal{D}^\rho \sigma \\ & - \frac{e^{2\sigma}}{4} V_{[\mu}\, h_{\nu]\lambda\rho} V^{\lambda\rho} + \frac{e^{-2\sigma}}{4} W_{[\mu}\, h_{\nu]\lambda\rho} W^{\lambda\rho}\Big] \ ,\end{aligned} \tag{1.66}$$

and both $g_{\mu\nu}$ and ϕ remain invariant. These match with the rules given by Kaloper and Meissner in [64] up to the sign of the α' correction of the b-field.[10]

1.5 Final discussion and conclusions

This chapter has been devoted to developing the tools needed to continue our discussion in subsequent sections of the first part of the thesis. Its most important results are the BdR theory, (1.31), together with the rules implementing corrected T-duality, (1.49). As already mentioned, we are particularly interested in this theory because it allows us to generalize the low-energy background dynamics of string theory, while preserving one of its most important and characteristic properties: T-duality invariance. This, in turn, sets the stage for asking ourselves relevant questions about the physics of these

[10] The fact that this is a typo in [64] is confirmed by the fact that their (4.9) and (4.11) are not compatible. For the field H of [64] (here h), which is even under T-duality at leading order in α', the correction to the T-duality transformation should rather be -2 the expression in (4.9). For odd fields the same contribution would be instead multiplied by $+2$. This easily follows from the first calculation they do to remove by a field redefinition the part of the action that is odd under T-duality, which is later reinterpreted as a correction to the T-duality transformation.

theories. In particular, given that T-duality relates two solutions which are generically very different – *e.g.*, it can change the asymptotic structure, as we will explicitly see in an example in chapter 3 –, does it change physical properties of the solution, like black hole thermodynamical quantities? We expect the answer in string theory to be no, because T-duality should be a complete physical equivalence. But, having at our disposal theories which are not related to any known low-energy string theory, we do not know what to expect for generic values of the perturbative parameters of the BdR theory. It could be that string theory values are singled out or, as we will show it happens, it could be that invariance of black hole entropy and temperature is guaranteed by the T-duality symmetric character of the action, irrespective of its string sigma model origin. This would indicate that T-duality provides physical equivalences even beyond the realm of string theory.

In addition, in (1.54) we have also presented T-duality rules in terms of dimensionally reduced fields for a broad set of schemes, meaning different redefinitions of the fields as in (1.53). These rules can be used to generate new solutions starting from known ones in a given scheme, so they are certainly a useful tool to analyze the solution landscape of low-energy string theory – or, more generally, of any theory related to the BdR one via field redefinitions. Connection with some popular schemes in the literature, like those in [64, 68], has also been provided.

The natural question is whether the construction we have presented in this chapter can be carried out to higher-orders in the perturbative expansion, which in string theory would imply including higher powers in the α' expansion. This is a technically complicated task, since already at first order manipulations are tedious, and the freedom provided by field redefinitions has to be taken into account. Therefore, we decided to stick to first order corrections, as they will prove to be enough to test the questions we will pose in the next chapter. Let us mention, however, that some interesting works pursuing the construction of higher-order corrected actions within the framework of DFT are available [79]. For certain kinds of backgrounds, in particular those relevant in cosmology, even an all-order construction has been developed [80, 81]. On the negative side, the very possibility of extending the general $O(D, D)$-invariant construction of low-energy string theory has been called into question recently [82].[11] All this shows that, even if not discussed in the present thesis, extending the construction of this chapter to higher-orders in the perturbative expansion is an interesting problem that deserves further study in the future.

[11] This does not mean that T-duality invariance of low-energy string effective actions can fail. DFT aims at making the duality-symmetry manifest before compactifying the background invariant directions, and this is not guaranteed to be possible. The compactified theory, on the contrary, is certainly T-duality invariant.

2

Black hole thermodynamics in the T-duality invariant effective theories

Black holes are, without a doubt, one of the most remarkable predictions of Einstein's theory of General Relativity. Initially overlooked during the early stages of the development of the theory in the first half of the twentieth century, they have consistently gained in relevance during the last decades, not only in the theoretical side but also with experimental results that point unambiguously towards their existence [12, 13]. One of the biggest puzzles that emerged from the theoretical study of black holes is the fact that they behave as thermodynamic objects. The laws of black hole thermodynamics were originally discovered as purely mechanical results, consequence of the theory of General Relativity [11], but their similarity with the basic laws of thermodynamics was soon observed [14]. Hawking's calculation of the thermal emission of particles by black holes [15] left no doubt about the thermodynamic nature of these gravitational beasts, and the microscopic origin of their entropy [16] has remained a mystery since then. It is fair to say that, despite many advances, we still lack a complete understanding of the microscopic origin of black hole entropy, and obtaining it would probably lead to a quantum theory of gravity.

What we have developed over time is a better and more diverse set of techniques to obtain the entropy of black holes in different geometrical theories of gravity. From the Gibbons-Hawking Euclidean methods [83] to the Wald's Noether charge picture [51, 84], we have currently several tools at our disposal to calculate the entropy of black holes in a given theory. The Noether charge approach pioneered by Wald incorporates in a very natural way the first law of black hole thermodynamics, and it can also be easily adapted to theories containing many types of corrections and extra fields, in addition to the metric. For these reasons, it will be our main focus in the present chapter, where we aim to derive the black hole entropy formula for the generalized BdR theory, (1.31). The presence of the vielbein as a basic object of the theory, instead of the metric, will force us to consider some particular modifications of the original Wald procedure, along the lines presented in previous works [58, 85].

All in all, our goal is to understand the way black hole thermodynamical quantities behave in the generalized BdR theory. In particular, we want to analyze whether black hole entropy and temperature are invariant under (corrected) T-duality transformations, which were argued to be a symmetry of the theory in the previous chapter. On the one hand, if these transformations map physically equivalent solutions, we would expect their thermodynamic properties to be equal. This must be certainly true for the values of the parameters coming from string theories: in those cases the sigma model origin guarantees the complete equivalence of a given background and its T-dual. In fact, this result was obtained for the leading order low-energy effective theory, (1.2), in [86]. In that work the invariance of black hole entropy was shown to be a consequence of its geometrical nature: it is given essentially by the black hole horizon area in Einstein frame, and leading-order T-duality transformations map horizons into horizons while preserving the area (the surface gravity, and therefore the temperature, are also preserved). But, on the other hand, when higher derivative corrections are included, we expect to lose the purely geometrical picture of the entropy. Furthermore, for arbitrary values of the parameters of the generalized BdR theory, we do not know of any sigma model to which we can attribute the origin of the low-energy theory. Being sensitive to microscopic degrees of freedom, black hole entropy is a quantity likely to discriminate between theories which come from a healthy sigma model and those which do not. This is the main motivation behind the study presented in this chapter. As we will show, the entropy and temperature of generic non-extremal black hole solutions of the generalized BdR theory are T-duality invariant, irrespective of the values of the free parameters in the theory. Thus, string theory low-energy models are not favored in any way by this test, and T-duality might actually provide complete physical equivalences even beyond the string realm.

2.1 Generalized Wald procedure: introduction

Conserved charges in gauge theories are a subtle issue. Wald's proposal to interpret the black hole entropy as a Noether charge associated with diffeomorphism symmetry [51, 84] involves discussing some of these subtleties, which is what we intend to do in this section. Notice that our goal is to compute the entropy of black hole spacetimes with a bifurcate Killing horizon in the BdR theory, (1.31), and that this theory is naturally written in a first order formalism in which the vielbein is a fundamental field. Thus, we are forced to consider local Lorentz transformations as a gauge symmetry. This implies that the original Wald procedure has to be slightly adapted, following the lines presented in [85]. Although we will not employ its full power here, a natural language to discuss conservation laws in geometric theories is the covariant phase space formalism. We refer the reader to the lectures [57] for further details.

Our starting point is a Lagrangian D-form[1] $\mathbf{L} = \boldsymbol{\epsilon}\,\mathcal{L}$ (with $\boldsymbol{\epsilon}$ the volume form) which, under a general variation, satisfies:

$$\delta\mathbf{L} = \mathbf{E_i}\,\delta\Psi^i + \mathrm{d}\boldsymbol{\theta}(\Psi,\delta\Psi)\ , \tag{2.1}$$

where $\Psi = \{\Psi^i\}$ stands for all of our fundamental fields, $\mathbf{E_i} = 0$ are the equations of motion and the second term is a total derivative. A presymplectic form can be defined via a second variation as:

$$\boldsymbol{\Omega}(\Psi,\delta_1\Psi,\delta_2\Psi) = \delta_1\boldsymbol{\theta}(\Psi,\delta_2\Psi) - \delta_2\boldsymbol{\theta}(\Psi,\delta_1\Psi)\ , \tag{2.2}$$

[1] Differential form language will be extensively used in this section. The conventions we adopt are set out in the Notations and conventions section.

where δ_1 and δ_2 are two generic and independent infinitesimal variations, which are to be interpreted as exterior derivatives in the space of fields [57]. This quantity will be relevant when deriving an explicit form of the first law. For the moment, let us consider generalized variations of the fields $\delta_\Gamma \Psi$ which are symmetries of the action. Γ represents the set of parameters of the transformation, containing at least a vector field ζ in a diffeomorphism invariant theory. To make things more concrete, let us look to a couple of well known examples. In General Relativity, where the metric is the only field present and diffeomorphisms are the gauge symmetry, we would take $\Gamma = \zeta$ and:

$$\delta_\zeta G_{MN} = \mathcal{L}_\zeta G_{MN} = 2\nabla_{(M}\zeta_{N)} \ . \tag{2.3}$$

If we considered Einstein-Maxwell theory, we would include another parameter β corresponding to the U(1) gauge transformation, so that $\Gamma = (\zeta, \beta)$. In addition to the previous transformation, we would also have:

$$\delta_\Gamma A_M = \mathcal{L}_\zeta A_M + \partial_M \beta \ . \tag{2.4}$$

For the action (1.31), we will have to take $\Gamma = (\zeta, \lambda, \beta)$, where λ and β are the parameters of the anomalous Lorentz and B-field gauge transformations, respectively (the precise form of the variations will be discussed below). As already mentioned, the transformation δ_Γ must be a symmetry of our theory, in the sense that

$$\delta_\Gamma \mathbf{L} = \mathcal{L}_\zeta \mathbf{L} + \mathrm{d}\mathbf{\Xi}_\Gamma = \mathrm{d}\left(i_\zeta \mathbf{L} + \mathbf{\Xi}_\Gamma\right) \ . \tag{2.5}$$

In this thesis, we will always consider exactly invariant Lagrangians, so that $\mathbf{\Xi}_\Gamma = 0$. There are cases in which this boundary term plays an important role, though, some examples can be found in [87]. Using (2.1) and (2.5), we can define the Noether current [84]:

$$\mathbf{j}_\Gamma = \boldsymbol{\theta}(\Psi, \delta_\Gamma \Psi) - i_\zeta \mathbf{L} \ , \tag{2.6}$$

whose exterior derivative vanishes on-shell, $\mathrm{d}\mathbf{j}_\Gamma \cong 0$ ($\cong$ stands for equality on-shell), thereby[2]

$$\mathbf{j}_\Gamma \cong \mathrm{d}\mathbf{Q}_\Gamma \ . \tag{2.7}$$

We need to study now the transformation law of $\boldsymbol{\theta}(\Psi, \delta\Psi)$ in order to obtain the abstract relation that in the context of black hole spacetimes will give rise to the first law of thermodynamics. In general, we write $\delta_\Gamma \boldsymbol{\theta}(\Psi, \delta\Psi)$ in the following form:

$$\delta_\Gamma \boldsymbol{\theta}(\Psi, \delta\Psi) = \mathcal{L}_\zeta \boldsymbol{\theta}(\Psi, \delta\Psi) + \mathbf{\Pi}_\Gamma(\Psi, \delta\Psi) \ , \tag{2.8}$$

where $\mathbf{\Pi}_\Gamma(\Psi, \delta\Psi)$ accounts for the non-covariant part – *i.e.*, not captured by the Lie derivative – of $\boldsymbol{\theta}(\Phi, \delta\Phi)$. Equating $\delta\delta_\Gamma \mathbf{L}$ with $\delta_\Gamma \delta \mathbf{L}$ and using the compatibility between exterior and Lie derivatives, we obtain $0 = \mathrm{d}\mathbf{\Pi}_\Gamma(\Psi, \delta\Psi)$. Thus, we can obtain a $(D-2)$-form $\mathbf{\Sigma}_\Gamma(\Psi, \delta\Psi)$ such that $\mathrm{d}\mathbf{\Sigma}_\Gamma(\Psi, \delta\Psi) = \mathbf{\Pi}_\Gamma(\Psi, \delta\Psi)$. Finally, applying δ to (2.6) – and

[2]A closed form that depends linearly in the gauge parameters and their derivatives is locally exact, as can be shown using the more powerful tools of the covariant phase space formalism, [57, 88]. With the same tools, one can show that $\mathbf{j}_\Gamma$ is not only closed on-shell, but in fact it exists a $(D-1)$-form $\mathbf{S}_\Gamma$ such that $\mathbf{j}_\Gamma - \mathbf{S}_\Gamma$ is closed off-shell and $\mathbf{S}_\Gamma$ vanishes on-shell. This is the technically correct step one should perform when defining the charge.

after some algebra – we can demonstrate that the presymplectic form evaluated on-shell reads:

$$\boldsymbol{\Omega}(\Psi, \delta\Psi, \delta_\Gamma\Psi) \cong \mathrm{d}\left[\delta\mathbf{Q}_\Gamma - i_\varsigma\boldsymbol{\theta}(\Psi, \delta\Psi) - \boldsymbol{\Sigma}_\Gamma(\Psi, \delta\Psi)\right] \ , \tag{2.9}$$

where we require not only the fields, but also the first order variations to be on-shell, *i.e.*, connecting two infinitesimally close solutions of the equations of motion. Defining $\mathbf{k}_\Gamma(\Psi, \delta\Psi) \equiv \delta\mathbf{Q}_\Gamma - i_\varsigma\boldsymbol{\theta}(\Psi, \delta\Psi) - \boldsymbol{\Sigma}_\Gamma(\Psi, \delta\Psi)$, where in the first term we are only varying the fields of our theory (and not the parameters Γ), we have that:

$$\boldsymbol{\Omega}(\Psi, \delta\Psi, \delta_\Gamma\Psi) \cong \mathrm{d}\mathbf{k}_\Gamma\left(\Psi, \delta\Psi\right) \ . \tag{2.10}$$

This can be understood as a conservation law for the charge $\mathbf{k}_\Gamma(\Psi, \delta\Psi)$ between two infinitesimally close field configurations provided that $\mathrm{d}\mathbf{k}_\Gamma(\Psi, \delta\Psi) \cong 0$. In order to guarantee this, we will restrict ourselves to symmetry transformations which vanish on-shell, $\delta_\Gamma\Psi \cong 0$. Being $\boldsymbol{\Omega}(\Psi, \delta\Psi, \delta_\Gamma\Psi)$ bilinear in the variations, this makes the left hand side of the previous equation equal to zero. To put it in more familiar terms, if we look at the Einstein gravity transformation, (2.3), we need to choose the vector field of the transformation in such a way that $\mathcal{L}_\varsigma G_{MN} \cong 0$. That is, we must choose a Killing vector of the background metric to obtain the conservation law. If we also have a U(1) field as in (2.4), the Killing field and the U(1) gauge parameter must be chosen to satisfy:

$$\delta_\Gamma A_M = \mathcal{L}_\varsigma A_M + \partial_M\beta \cong 0 \ . \tag{2.11}$$

Usually, the U(1) background field is also invariant under the Killing flow, and then we must choose $\beta = c \in \mathbb{R}$ to satisfy the previous equation.

What are the symmetry transformations of our action, (1.31)? As already mentioned, we will have a vector field parametrizing diffeomorphisms, a local (antisymmetric) Lorentz parameter, and a one-form for the gauge transformations of the B-field. Imposing the conditions to have on-shell vanishing transformations, we take the corresponding parameters to be $(\xi, \lambda^E_\xi, \alpha_\xi)$ such that:[3]

$$\delta_\xi\Phi = \mathcal{L}_\xi\Phi \cong 0 \ , \tag{2.12a}$$

$$\delta_\xi E^A = \mathcal{L}_\xi E^A + E^B(\lambda^E_\xi)_B{}^A \cong 0 \ , \tag{2.12b}$$

$$\delta_\xi B = \mathcal{L}_\xi B - \frac{a_-}{4}\mathrm{d}(\lambda^E_\xi)_A{}^B \wedge \Omega^{(-)A}_B + \frac{a_+}{4}\mathrm{d}(\lambda^E_\xi)_A{}^B \wedge \Omega^{(+)A}_B + \mathrm{d}\alpha_\xi \cong 0 \ . \tag{2.12c}$$

In the same way as we did in the more familiar examples, the parameters of the transformation must be chosen to ensure that the on-shell identities are satisfied, $\delta_\xi\Psi \cong 0$ for all the fields Ψ. This can be done in the following way. We take ξ to be a Killing vector such that the dilaton is also invariant under its flow, to satisfy the first of the previous equations. [85] showed that, in this case, the on-shell invariance of the vielbein is ensured provided that we take:

$$(\lambda^E_\xi)^{AB} \equiv \mathcal{L}_\xi(E^{[A})_S(E^{B]})^S \ . \tag{2.13}$$

We finally take the parameter of the B-field gauge transformation, α_ξ, in such a way that the last of the previous transformations vanishes. Most times, this field will also be

[3]Note that we have a change of sign with respect to [85] in the definition of $(\lambda^E_\xi)^{AB}$, due to the different conventions used for Lorentz transformations. Also, we will use differential form notation for the B-field and other objects such as the spin connection when convenient, although we will not explicitly write boldface symbols. Based on the context, it will be easy to determine when we do so.

invariant under the Killing flow, and for a suitable choice of vielbein the parameter (λ_ξ^E) is such that we can take α_ξ to be a closed form to guarantee $\delta_\xi B \cong 0$. We will keep it general for the moment, though.

After this small digression clarifying what the symmetry transformations are in some familiar situations as well as in the BdR theory, let us go back to equation (2.10). We will apply it in the context of a black hole spacetime having a bifurcate Killing horizon – which is the general situation for a non-extremal black hole in equilibrium, [89]. This means we have a Killing field generating the horizon, ξ, and we will take this to be the field ζ in the general variation parametrized by Γ. Furthermore, notice that ξ vanishes at the bifurcation surface, which we denote by $\mathcal{B}$. Integrating then (2.10) on a hypersurface with boundaries at $\mathcal{B}$ and at asymptotic infinity we obtain:

$$\int_{\mathcal{B}} \mathbf{k}_\xi(\Psi, \delta\Psi) \cong \int_\infty \mathbf{k}_\xi(\Psi, \delta\Psi) \ . \tag{2.14}$$

This is the fundamental result behind the first law of black hole thermodynamics. Essentially, it is telling us that the variation of certain charges at the horizon between two infinitesimally close black hole solutions coincides with the variation of other charges, defined at infinity. What charges are included at each of the surfaces is a subtle issue. In General Relativity, the variation of the mass and the angular momentum of the black hole appear in the right hand side of (2.14). The bifurcation surface integral produces then the entropic part of the first law, $\kappa\delta A_H/(8\pi G_D)$, κ being the surface gravity of the horizon and A_H its area.[4] In Einstein-Maxwell theory, the charge term of the first law can appear either at the bifurcation surface as $\Phi_{\rm EM}\delta Q_{\rm EM}$ if the field A_M is not regular there, or at infinity as $-\Phi_{\rm EM}\delta Q_{\rm EM}$ if we impose regularity at $\mathcal{B}$. This is discussed in detail in [90], where it is also shown that the difference between these situations is only a gauge transformation of the U(1) field. This is in principle general: if the fields are all regular at $\mathcal{B}$, which we will require, all the charges (energy, angular momentum, gauge charges, ...) appear as part of the integral at infinity. In this situation, the left hand side of (2.9) is $T_H\delta S$, which allows us to identify the temperature and entropy of the black hole as:

$$T_H = \frac{\kappa}{2\pi} \ , \qquad \delta S = \frac{2\pi}{\kappa} \int_{\mathcal{B}} \mathbf{k}_\xi(\Psi, \delta\Psi)\Big|_{\xi\to 0, \nabla_M\xi_N \to \kappa\, n_{MN}} \ , \tag{2.15}$$

where we have employed $\xi|_{\mathcal{B}} = 0$ and $\nabla_M\xi_N|_{\mathcal{B}} = \kappa\, n_{MN}$, n_{MN} being the binormal to $\mathcal{B}$ and taking ξ to be properly normalized [58, 85].

The variation δS can be written in a different form under some extra assumptions. First of all, ξ vanishes at the bifurcation surface, so the term $i_\xi\boldsymbol{\theta}(\Psi, \delta\Psi)$ does not contribute to the integral in $\mathcal{B}$ if our fields are all regular. Furthermore, in the BdR action $\boldsymbol{\theta}(\Psi, \delta\Psi)$ is such that $\boldsymbol{\Sigma}_\xi(\Psi, \delta\Psi)$ has no relevant contribution at the bifurcation surface on-shell. In this situation:

$$\delta S = \frac{2\pi}{\kappa}\, \delta \int_{\mathcal{B}} \mathbf{Q}_\xi(\Psi)\Big|_{\xi\to 0, \nabla_M\xi_N \to \kappa\, n_{MN}} \ , \tag{2.16}$$

where $\mathbf{Q}_\xi$ was introduced in (2.7). Finally, since terms linear in ξ in the integral will not contribute at the bifurcation surface, we find that the relevant contribution in $\mathbf{Q}_\xi(\Psi)$

[4] This computation can be found with full level of detail in [58].

is linear in $\nabla_M \xi_N$, and thus linear in κ when evaluated at $\mathcal{B}$. The surface gravity κ is constant (zeroth law), and $\delta\kappa = 0$, understanding δ as a variation leaving the Killing field ξ fixed [51, 84]. As a consequence, under the previous assumptions we obtain an expression for the entropy as an integral over the bifurcation surface:

$$S = 2\pi \int_{\mathcal{B}} \mathbf{Q}_\xi(\Psi)\Big|_{\xi\to 0, \nabla_M \xi_N \to n_{MN}} . \tag{2.17}$$

This expression, combined with the definition of the proper on-shell vanishing symmetry transformations (2.12), are the tools we need to compute the entropy of the full BdR theory, (1.31). We will present the results in the next section.

2.2 Black hole entropy in the generalized BdR theory

Consider the generalized BdR action, (1.31), and split it as $\mathcal{I}_{\mathrm{BdR}} = (\mathcal{I}_0 + \mathcal{I}_{H'^2} + \mathcal{I}_{R^2})/(2\kappa_s^2)$, where we define:

$$\mathcal{I}_0 \equiv \int \boldsymbol{\epsilon}\, e^{-2\Phi} \left[R - 2\Lambda + 4\,(\nabla\Phi)^2\right] , \tag{2.18a}$$

$$\mathcal{I}_{H'^2} \equiv -\frac{1}{12}\int \boldsymbol{\epsilon}\, e^{-2\Phi} H'_{MNR} H'^{MNR} = -\frac{1}{2}\int e^{-2\Phi} \star H' \wedge H' , \tag{2.18b}$$

$$\mathcal{I}_{R^2} \equiv \sum_{k=\pm} \frac{a_k}{8} \int \boldsymbol{\epsilon}\, e^{-2\Phi} R^{(k)}_{MNA}{}^{B} R^{(k)MN}{}_{B}{}^{A} = \sum_{k=\pm} \frac{a_k}{4} \int e^{-2\Phi} \star R^{(k)B}_{A} \wedge R^{(k)A}_{B} . \tag{2.18c}$$

Notice that we have performed an innocuous integration by parts in $\mathcal{I}_0$ in order to obtain a more convenient form of the dilaton kinetic term, and differential form notation has been used – with $\boldsymbol{\epsilon}$ the volume form. The factor $(2\kappa_s^2)^{-1}$ has also been extracted from the previous definitions to simplify the expressions, we will restore it at the end of the computation. In the following we will explicitly compute the entropy associated with $\mathcal{I}_0$, following the steps outlined in the previous section. The corresponding results for $\mathcal{I}_{H'^2}$ and $\mathcal{I}_{R^2}$ will only be quoted here in the main text, leaving the details of the computation to appendix A.

2.2.1 Leading order entropy

We must start by varying the Lagrangian associated with the action $\mathcal{I}_0$ to compute the boundary term, $\boldsymbol{\theta}(\Psi, \delta\Psi)$. Using the fact that, under a general variation of the fields, $\delta\boldsymbol{\epsilon} = \frac{1}{2} G^{MN} \delta G_{MN} \boldsymbol{\epsilon}$,

$$\begin{aligned}\delta \mathbf{L}_0 = \boldsymbol{\epsilon}\, e^{-2\Phi} \Big[&-2\mathcal{L}_0\, \delta\Phi + 8\nabla^M\Phi\nabla_M\delta\Phi + \nabla_M X^M[\delta G] \\ &+ \left(-R^{MN} - 4\nabla^M\Phi\nabla^N\Phi + \frac{1}{2}G^{MN}\mathcal{L}_0\right)\delta G_{MN}\Big] ,\end{aligned} \tag{2.19}$$

where $\mathbf{L}_0 = \boldsymbol{\epsilon}\,\mathcal{L}_0$, and $X^M[\delta G] = G^{PQ}\delta\Gamma^M_{PQ} - G^{MP}\delta\Gamma^Q_{PQ}$. Terms with $\delta\Phi$ or δG_{MN} will be part of the equations of motion (the other parts coming from $\mathcal{I}_{H'^2}$ and $\mathcal{I}_{R^2}$). We can thus forget about them for the boundary term $\boldsymbol{\theta}(\Psi, \delta\Psi)$. To simplify the remaining terms, we have to take into account the symmetry transformations δ_Γ we employ to obtain the entropy charge. Considering diffeormophisms, anomalous Lorentz invariance, and the gauge symmetry of the B field, the following are the symmetry transformations of the fundamental fields for generic parameters:

$$\delta_\Gamma \Phi = \mathcal{L}_\zeta \Phi \ , \tag{2.20a}$$

$$\delta_\Gamma E^A = \mathcal{L}_\zeta E^A + E^B \lambda_B{}^A \ , \tag{2.20b}$$

$$\delta_\Gamma B = \mathcal{L}_\zeta B + \gamma_- \Omega_A{}^B \wedge \mathrm{d}\lambda_B{}^A + \frac{\gamma_+}{2} H_A{}^B \wedge \mathrm{d}\lambda_B{}^A + \mathrm{d}\beta \ . \tag{2.20c}$$

There are two differences with respect to the transformations presented in (2.12). First of all, we have rewritten the B-field transformation by splitting the torsionful connections according to their definition, (1.28), and then changing the parameters $a_\pm$ by $\gamma_\pm$, (1.55). We have also used differential form notation, where for the H-field with two flat indices $H_A{}^B \equiv H_{MA}{}^B \mathrm{d}x^M$. But the main difference is in the fact that we are not imposing the previous variations to vanish on-shell, as que did in (2.12). This is the important difference between δ_Γ (a general gauge symmetry transformation of our theory, parameterized by ζ, λ, and β) and δ_ξ (the specific transformation that vanishes on-shell). Of course, given a background solution, we can set $\zeta = \xi$ (with ξ being a Killing field), $\lambda = \lambda_\xi^E$ as in (2.13), and $\beta = \alpha_\xi$, and we recover (2.12). But, for the moment, δ_Γ is a generic symmetry transformation. It will be useful to have also the transformation of the remaining fields:

$$\delta_\Gamma G_{MN} = \mathcal{L}_\zeta G_{MN} \ , \tag{2.21a}$$

$$\delta_\Gamma \Omega_A{}^B = \mathcal{L}_\zeta \Omega_A{}^B + \mathrm{d}\lambda_A{}^B + \Omega_A{}^C \lambda_C{}^B - \lambda_A{}^C \Omega_C{}^B \ , \tag{2.21b}$$

$$\delta_\Gamma H_A{}^B = \mathcal{L}_\zeta H_A{}^B - \lambda_A{}^C H_C{}^B + H_A{}^C \lambda_C{}^B + \mathcal{O}(a_\pm) \ . \tag{2.21c}$$

Notice that $\mathcal{O}(a_\pm)$ means that we only write the leading order part of the transformation of $H_A{}^B$. This will be enough for our purposes.

In order to compute the contribution to the black hole entropy, the first step is to find the charge $\mathbf{Q}_\Gamma$ in terms of generic transformation parameters (ζ, λ, β). Then we simply substitute those parameters by $(\xi, \lambda_\xi^E, \alpha_\xi)$, which make the previous variations to vanish on-shell. In order to simplify some of the computations, it is important to keep in mind that at the end of the day the charge has to be evaluated on the bifurcation surface $\mathcal{B}$, and also that $\xi^M|_\mathcal{B} = 0$. For this reason, terms in $\mathbf{Q}_\xi$ which are linear in ξ vanish at the bifurcation surface and will not contribute to the entropy integrand, (2.17).[5] Since we obtain the charge by doing two integrations by parts on $\delta\mathbf{L}$, the terms of $\mathbf{Q}_\xi$ with $\nabla\xi$ come from those of $\delta\mathbf{L}$ with three or more derivatives. Other terms with less than three derivatives are not relevant for the entropy and will be ignored in the derivations that follow. This derivative-counting procedure to simplify the computations taking advantage of the behaviour of the Killing field in $\mathcal{B}$ is a common technique in black hole entropy discussions, see for example [91]. Going back to our expression (2.19), $\nabla_M \delta_\xi \Phi$ has at most two derivatives of the vector field, and is therefore irrelevant. However, the term with $X^M[\delta G]$ will be relevant, and thus we are left with:

$$\delta\mathbf{L}_0 = \boldsymbol{\epsilon}\, e^{-2\Phi} \nabla_M X^M[\delta G] + \ldots = \boldsymbol{\epsilon}\, \nabla_M \left(e^{-2\Phi} X^M[\delta G]\right) + \ldots \tag{2.22}$$

[5]For this to be true it is essential that all fields are regular at $\mathcal{B}$. We are imposing this as a requirement for our computation to be valid.

This is the relevant part of $\mathrm{d}\boldsymbol{\theta}_0(\Psi, \delta\Psi)$, the dots denote all the remaining terms in the variation of the Lagrangian which will not be important for the entropy computation (either because they are part of the equations of motion, or because its contribution to $\mathbf{Q}_\xi$ vanishes at $\mathcal{B}$). Using the duality between p-forms and $(D-p)$-vectors we can easily read:[6]

$$\theta_0^M(\Psi, \delta\Psi) = 2e^{-2\Phi} G^{P[Q} \delta\Gamma^{M]}_{PQ} + \ldots = 2e^{-2\Phi} G^{MN} G^{PQ} \nabla_{[P} \delta G_{N]Q} + \ldots \tag{2.23}$$

It is now a simple matter to construct the current $j^M_{0,\Gamma} = \theta^M_0(\Psi, \delta_\Gamma \Psi) - \zeta^M e^{-2\Phi} \mathcal{L}_0$. Keeping only the relevant terms, it is given by:

$$\begin{aligned} j^M_{0,\Gamma} &= e^{-2\Phi} G^{MN} G^{PQ} \left(\nabla_P \nabla_Q \zeta_N - \nabla_N \nabla_Q \zeta_P\right) + \ldots \\ &= e^{-2\Phi} G^{MN} G^{PQ} \left(\nabla_P \nabla_Q \zeta_N - 2\nabla_{[N} \nabla_{Q]} \zeta_P - \nabla_Q \nabla_N \zeta_P\right) + \ldots \\ &= e^{-2\Phi} \nabla_P \left(\nabla^P \zeta^M - \nabla^M \zeta^P\right) + \ldots \\ &= 2\nabla_N \left(e^{-2\Phi} \nabla^{[N} \zeta^{M]}\right) + \ldots \end{aligned} \tag{2.24}$$

Notice the use of the Ricci identity in the second line to discard one of the terms. This is already in a suitable form to read the associated charge; using $\nabla_N Q^{MN}_{0,\Gamma} = j^M_{0,\Gamma}$, it is immediate to conclude:[7]

$$\mathbf{Q}_{0,\Gamma} = -2e^{-2\Phi} \nabla^M \zeta^N \left(\mathrm{d}^{D-2} x\right)_{MN} + \ldots \tag{2.25}$$

Using (2.17) it is immediate to write the contribution to the entropy formula of the leading order part of the action, $\mathcal{I}_0$. We evaluate the previous charge for the set of gauge parameters fulfilling (2.12), $(\xi, \lambda^e_\xi, \alpha_\xi)$, where ξ is the horizon-generating Killing field, obtaining:

$$S_0 = 2\pi \int_{\mathcal{B}} \mathbf{Q}_{0,\xi} \Big|_{\xi \to 0, \nabla_M \xi_N \to n_{MN}} = -4\pi \int_{\mathcal{B}} e^{-2\Phi} n^{MN} \left(\mathrm{d}^{D-2} x\right)_{MN} = 4\pi \int_{\mathcal{B}} e^{-2\Phi} \bar{\boldsymbol{\epsilon}} \ , \tag{2.26}$$

where recall that $\left(\mathrm{d}^{D-2} x\right)_{MN} |_{\mathcal{B}} = n_{MN} \bar{\boldsymbol{\epsilon}}/2$, $\bar{\boldsymbol{\epsilon}}$ is the induced volume form in $\mathcal{B}$, and $n_{MN} n^{MN} = -2$ is the normalization of the binormal. We have thus arrived at the expected contribution for the leading order part of the entropy: it is nothing but the usual area-law Bekenstein-Hawking entropy corrected by the dilaton term, as a consequence of being working with string frame fields.

2.2.2 First order corrections

The previous computation of S_0 helps to get a taste of the kind of computations we need to perform to derive the black hole entropy formula. The contributions from $\mathcal{I}_{H'^2}$ and $\mathcal{I}_{R^2}$ are technically more difficult to derive, and for this reason we relegate the explicit computations to appendix A. We will only quote here the main results. The full charge derived from the generalized BdR action is:

$$\mathbf{Q}_\xi = \frac{1}{2\kappa_s^2} \left(\mathbf{Q}_0 + \mathbf{Q}_{H'^2} + \mathbf{Q}_{R^2} + \mathbf{Q}_{\alpha_\xi}\right) \ , \tag{2.27}$$

[6]Up to the addition of a closed form to $\boldsymbol{\theta}(\Psi, \delta\Psi)$, which does not modify the entropy [91].

[7]Again, the primitive is defined up to closed form, but this ambiguity does not alter the entropy result [91].

where $\mathbf{Q}_0$ is the leading order term, obtained from (2.25) by setting the vector field of the transformation to the Killing ξ, and the remaining contributions are:

$$\mathbf{Q}_{H'^2} \equiv e^{-2\Phi} \star H \wedge \left(2\gamma_- \Omega_A{}^B + \gamma_+ H_A{}^B\right) (\lambda_\xi^E)_B{}^A + \dots \ , \tag{2.28a}$$

$$\mathbf{Q}_{R^2} \equiv -e^{-2\Phi} \left[2\gamma_+ \star \left(R_A{}^B + \frac{1}{4} H_A{}^C \wedge H_C{}^B\right) \right. \\ \left. + \gamma_- \star \left(\mathrm{d}H_A{}^B + 2\Omega_A{}^C \wedge H_C{}^B\right)\right] (\lambda_\xi^E)_B{}^A + \dots \ , \tag{2.28b}$$

$$\mathbf{Q}_{\alpha_\xi} \equiv 6\mathbb{E}^{MNR} (\alpha_\xi)_R \left(\mathrm{d}^{D-2}x\right)_{MN} \ , \tag{2.28c}$$

where we have defined $\mathbb{E}^{MNR} \equiv T^{MNR} - \nabla_Q S^{QMNR}$, with:

$$T^{MNR} \equiv \frac{\partial \mathcal{L}}{\partial H_{MNR}} \ , \qquad S^{QMNR} \equiv \frac{\partial \mathcal{L}}{\partial \nabla_Q H_{MNR}} \ . \tag{2.29}$$

Notice that $\mathcal{L}$ is defined to be the Lagrangian scalar $\mathbf{L} = \boldsymbol{\epsilon}\, \mathcal{L}$, and the B-field equation of motion can be written as $\nabla_M \mathbb{E}^{MNR} \cong 0$. Just like in the leading order derivation, dots in the entropy charges denote omitted terms which do not contribute when evaluated at the bifurcation surface (that is, terms proportional to ξ^M, thereby vanishing from the assumption of regularity applied to all fields). Finally, it is important to remember that α_ξ is not a free parameter of a gauge transformation. It is determined (up to the addition of a closed form) from the condition that the variation of the B-field given by (2.12) has to vanish on-shell. Later on we will see that we can make a vielbein choice to set $\alpha_\xi = 0$ in a region near the horizon but, for the moment, let us keep track of α_ξ . It will be necessary in order to show the invariance of the entropy under anomalous Lorentz transformations.

From the previous charges we can immediately read the corresponding entropy contributions. S_0 was already presented in (2.26), and the remaining terms are:

$$S_{H'^2} = 4\pi \int_{\mathcal{B}} e^{-2\Phi} \star H \wedge \left(\gamma_- \Omega^{AB} + \frac{\gamma_+}{2} H^{AB}\right) n_{AB} \ , \tag{2.30a}$$

$$S_{R^2} = -4\pi\gamma_+ \int_{\mathcal{B}} e^{-2\Phi} \star \left(R^{AB} + \frac{1}{4} H^{AC} \wedge H_C{}^B\right) n_{AB} \ , \tag{2.30b}$$

$$S_{\alpha_\xi} = 6\pi \int_{\mathcal{B}} \bar{\boldsymbol{\epsilon}}\, \mathbb{E}^{MNR} n_{MN} (\alpha_\xi)_R \ . \tag{2.30c}$$

We can further simplify and combine the $\gamma_+ HHn$ terms in the first two contributions. To do so, we write the fields in tensorial form, and we use the fact that the binormal can always be written as $n_{MN} = 2v_{[M} w_{N]}$ for some one-forms v, w [58]. This implies it must obey $n_{M[N} n_{RS]} = 0$, and then we have as our final result for the full entropy:

$$S = \frac{\pi}{\kappa_s^2} \int_{\mathcal{B}} \bar{\boldsymbol{\epsilon}}\, e^{-2\Phi} \left[2 - \gamma_+ \left(R^{MNRS} - \frac{3}{4} H^{TMN} H_T{}^{RS}\right) n_{MN} n_{RS} \right. \\ \left. + \gamma_- H^{TMN} \Omega_T{}^{RS} n_{MN} n_{RS}\right] + \frac{1}{2\kappa_s^2} S_{\alpha_\xi} \ , \tag{2.31}$$

where $\Omega_T{}^{RS} = \Omega_T{}^{AB} E_A{}^R E_B{}^S$. This is the final form of the entropy in the generalized BdR theory.

2.2.3 Anomalous Lorentz invariance of the entropy

Given that our theory is invariant under anomalous Lorentz transformations (1.36), this symmetry must be present in the entropy as well. Let us check this explicitly by considering the following infinitesimal transformation to a new set of fundamental fields:

$$E'^A = E^A + E^B \Lambda_B{}^A \ , \qquad\qquad \Phi' = \Phi \ , \tag{2.32a}$$

$$B' = B + \gamma_- \Omega_A{}^B \wedge \mathrm{d}\Lambda_B{}^A + \frac{\gamma_+}{2} H_A{}^B \wedge \mathrm{d}\Lambda_B{}^A \ . \tag{2.32b}$$

The new $(\lambda_\xi^{E'})$ for this vielbein becomes:

$$(\lambda_\xi^{E'})_B{}^A = (\lambda_\xi^E)_B{}^A + (\lambda_\xi^E)_B{}^C \Lambda_C{}^A - \Lambda_B{}^C (\lambda_\xi^E)_C{}^A - \mathcal{L}_\xi \Lambda_B{}^A \ . \tag{2.33}$$

Now, for these Lorentz transformed fields we must be sure that the symmetry transformations we employ to compute the entropy, (2.12), vanish on-shell. The new transformations are related to the old ones by:

$$\delta_\xi E'^A = \delta_\xi E^A + (\delta_\xi E^B)\Lambda_B{}^A \ , \tag{2.34a}$$

$$\begin{aligned} \delta_\xi B' = {} & \delta_\xi B + \gamma_- \delta_\xi \Omega_A{}^B \wedge \mathrm{d}\Lambda_B{}^A \\ & + \frac{\gamma_+}{2}\delta_\xi H_A{}^B \wedge \mathrm{d}\Lambda_B{}^A + \mathrm{d}\left[\delta_\Lambda \alpha_\xi - 2\gamma_- (\lambda_\xi^E)_A{}^B \mathrm{d}\Lambda_B{}^A\right] \ , \end{aligned} \tag{2.34b}$$

where $\delta_\Lambda \alpha_\xi = \alpha'_\xi - \alpha_\xi$. It follows from $\delta_\xi \Psi = 0$ for all fields Ψ that $\delta_\xi \Omega_A{}^B = 0$ and $\delta_\xi H_A{}^B = 0$. Consequently, we need $\mathrm{d}[\delta_\Lambda \alpha_\xi - 2\gamma_-(\lambda_\xi^E)_A{}^B \mathrm{d}\Lambda_B{}^A] = 0$ in order to satisfy $\delta_\xi B' = 0$. We choose $\delta_\Lambda \alpha_\xi = 2\gamma_-(\lambda_\xi^E)_A{}^B \mathrm{d}\Lambda_B{}^A$; that is, the choice of the suitable gauge parameter α_ξ must generically be changed under anomalous Lorentz transformation in order to guarantee $\delta_\xi B' = 0$. Let us go back to our computation of the anomalous Lorentz invariance of the entropy. Given the fact that, to first order, the only non-Lorentz covariant terms in the entropy (2.31) are S_{α_ξ} and the one containing the spin-connection, we conclude that:

$$\begin{aligned} \delta_\Lambda S &= \frac{\pi}{\kappa_s^2}\int_{\mathcal{B}} \bar{\boldsymbol{\epsilon}}\, e^{-2\Phi}\left[\gamma_- H^{MNR}\delta_\Lambda \Omega_R{}^{AB} n_{AB} - \frac{1}{2}H^{MNR}\delta_\Lambda(\alpha_\xi)_R\right] n_{MN} \\ &= \frac{\pi\gamma_-}{\kappa_s^2}\int_{\mathcal{B}} \bar{\boldsymbol{\epsilon}}\, e^{-2\Phi} H^{MNR}\partial_R \Lambda^{AB}\left[n_{AB} + (\lambda_\xi^E)_{AB}\right] n_{MN} = 0 \ , \end{aligned} \tag{2.35}$$

where we have used $(\lambda_\xi^E)_{AB}|_{\mathcal{B}} = -n_{AB}$ as proved in [85], the fact that $S^{QMNR} = \mathcal{O}(\gamma_\pm)$, and

$$T^{MNR} = -\frac{1}{6}e^{-2\Phi}H^{MNR} + \mathcal{O}(\gamma_\pm) \ . \tag{2.36}$$

As a consequence of these results, $\mathbb{E}^{MNR} = -e^{-2\Phi}H^{MNR}/6 + \mathcal{O}(\gamma_\pm)$, and the leading order value of $\mathbb{E}^{MNR}$ is enough to substitute in the variation of S_{α_ξ} because $\delta_\Lambda \alpha_\xi$ is already first order in γ_-. Thus, we conclude that the entropy is invariant under infinitesimal anomalous Lorentz transformations around a generic vielbein.

In proving this entropy invariance, the parameter α_ξ has played a key role. As presented when discussing the generalized Wald procedure, one needs to impose an invariant stationarity condition like $\delta_\xi \Psi = 0$ on the fields, and for the B-field this is only possible by means of α_ξ and its non-trivial anomalous Lorentz transformation. Let us look at this from a slightly different perspective. The condition $\delta_\xi B = 0$ is a statement about the existence of a diffeomorphism transformation which leaves a given background solution invariant, much in the same way $\delta_\xi G = 0$ is just the statement that there exists a Killing field leaving the metric invariant. The entropy in the generalized Wald construction is the charge associated with this transformation, derived by the methods of the previous section. The point with gauge fields, such as the B-field, is that we should look at different configurations as physically equivalent when they are related by a gauge transformation. Thus, the condition $\delta_\xi B = 0$ should be gauge invariant: a given B-field and a gauge-transformed version B' should satisfy $\delta_\xi B = \delta_\xi B'$ for the condition $\delta_\xi B = 0$ to be physically meaningful. In fact, this requirement for the vielbein when considering gauge invariance under local Lorentz transformations is what led [85] to define the transformation $\delta_\xi E^A$ with the particular value of λ_ξ^E given by (2.13). This makes the variation $\delta_\xi E^A$ covariant under the local Lorentz group. This idea of defining the diffeomorphism-related transformations which lead to the first law (and therefore to the entropy *à la Wald*) in an invariant way under the full gauge group of the theory has been developed with great level of detail in the past couple of years, shedding light on the issue of how the first law and the entropy are obtained when we have gravity plus extra gauge symmetries. Following the ideas of [92], the introduction of momentum maps in the works [93–95] has produced a framework in which entropy computations just like the one we have just presented can be done, without the appearance of parameters like our α_ξ which are not written in a closed form in terms of the background fields. We refer the reader to those works for further details, for our purposes our more rudimentary derivation of the entropy formula, (2.31), will be enough.

2.3 T-duality invariance of the entropy and temperature

In this section, we will discuss the properties of black hole entropy and temperature in the generalized BdR theory under the corrected T-duality transformations of the previous chapter. We will find that, to linear order in the perturbative parameters $a_\pm$, both quantities are invariant. This is true for all values of the parameters, even those not corresponding to effective string theories. Since the proof will be quite technical, it is worth discussing with some level of detail the exact setup we assume, and proving some axuliary results needed for the general argument.

2.3.1 Preliminaries: coordinates and vielbein near the horizon

We will deal with horizons of the kind described in [89]. Their main characteristic is that they are bifurcate Killing horizons in a stationary spacetime. Every regular Killing horizon with constant surface gravity $\kappa \neq 0$ is of bifurcate type and viceversa; we can take $\kappa > 0$ without loss of generality. These horizons can be extended to include a regular bifurcation surface $\mathcal{B}$, where we will evaluate the entropy. It is very convenient to use a generalization of the Kruskal coordinates for the region close to the horizon. As in the Schwarzschild black hole, they cover smoothly an entire neighborhood of the horizon, and in particular the bifurcation surface. The general line element in any spacetime dimension reads [89]:

$$\mathrm{d}s^2 = N\,\mathrm{d}U\mathrm{d}V + VF_\alpha\,\mathrm{d}U\mathrm{d}x^\alpha + \gamma_{\alpha\beta}\,\mathrm{d}x^\alpha\mathrm{d}x^\beta\ , \tag{2.37}$$

where N, F_α, and $\gamma_{\alpha\beta}$ are regular functions. Index notation starts to be a little bit cumbersome here, so let us review our conventions. The full D-dimensional spacetime has curved indices $M, N, \ldots$; and flat indices $A, B, \ldots$ for the vielbein. These are separated in two ways. First of all, for the T-duality transformations, the relevant splitting is the one presented in chapter 1: we take indices $\mu, \nu, \ldots$ (and $a, b, \ldots$ are the corresponding flat indices) in $(D-1)$-dimensional space, and we singularize the U(1) symmetric coordinate ψ (with corresponding flat index ι). But for the discussion of the present section, it is important to separate the lightcone coordinates (U, V) – collectively denoted by indices $\mu', \nu', \ldots$ – from the remaining $D-2$ tangent to the bifurcation surface $U = V = 0$, which will be indexed by $\alpha, \beta, \ldots$. These include ψ. Finally, we split this set of $D-2$ coordinates into ψ and the remaining $D-3$, indexed by $\alpha', \beta', \ldots$. The corresponding flat indices will be ι, just like before, and $a', b', \ldots$.

The null Killing field in the previous coordinates is given by $\xi = \kappa(U\partial_U - V\partial_V)$, where κ is the surface gravity with respect to ξ.[8] The U(1) symmetry along ψ imposes $\partial_\psi G_{MN} = 0$, and the same holds for $N, F_\alpha, \gamma_{\alpha\beta}$. Now we choose a vielbein for (2.37) as:[9]

$$E^0 = \frac{1}{\sqrt{2}}\left[\left(\frac{1}{2} + \frac{1}{4}V^2 e^{-2\sigma}F_\psi^2\right)\mathrm{d}U - N\,\mathrm{d}V + V\left(e^{-2\sigma}F_\psi\gamma_{\psi\alpha'} - F_{\alpha'}\right)\mathrm{d}x^{\alpha'}\right]\ , \tag{2.38a}$$

$$E^1 = \frac{1}{\sqrt{2}}\left[\left(\frac{1}{2} - \frac{1}{4}V^2 e^{-2\sigma}F_\psi^2\right)\mathrm{d}U + N\,\mathrm{d}V - V\left(e^{-2\sigma}F_\psi\gamma_{\psi\alpha'} - F_{\alpha'}\right)\mathrm{d}x^{\alpha'}\right]\ , \tag{2.38b}$$

$$E^{a'} = \mathrm{d}x^{\alpha'}e_{\alpha'}{}^{a'}\ ,\quad a' = 2, \ldots, D-2\ , \tag{2.38c}$$

$$E^\iota = \frac{1}{2}Ve^{-2\sigma}F_\psi\,\mathrm{d}U + e^{-\sigma}\gamma_{\psi\alpha'}\mathrm{d}x^{\alpha'} + e^\sigma\mathrm{d}\psi\ , \tag{2.38d}$$

where $\gamma_{\psi\psi} = e^{2\sigma}$, and the $e_{\alpha'}{}^{a'}$ constitute a vielbein for $\gamma_{\alpha'\beta'}$; *i.e.*, $\delta_{a'b'}e^{a'}e^{b'} = \gamma_{\alpha'\beta'}\,\mathrm{d}x^{\alpha'}\mathrm{d}x^{\beta'}$. This vielbein choice is convenient for three reasons. The first is that it contains $\mathrm{d}\psi$ only in the component E^ι and therefore it is of the form (1.9); consequently, the corresponding uncorrected T-duality simple rules of the previous chapter – (1.6) and (1.48) – apply to it. The second is that all components are smooth, and so is the inverse vielbein, $E_A{}^M$. Therefore, the connection components $\Omega_{MA}{}^B$ are regular as well, even on the bifurcation surface $\mathcal{B}$. Notice that regularity is crucial in the derivation of the entropy formula of the previous section, as already emphasized there (otherwise the derivative counting arguments might fail, [85]). Notice also that in this vielbein the stationarity condition $\delta_\xi E_M{}^A = \mathcal{L}_\xi E_M{}^A + E_M{}^B(\lambda_\xi^E)_B{}^A = 0$ is fulfilled with $(\lambda_\xi^E)_B{}^A$ being the generator of a uniform boost along the E^1 direction. This is a consequence of:

$$\mathcal{L}_\xi E_M{}^0 = \kappa E_M{}^1\ ,\qquad \mathcal{L}_\xi E_M{}^1 = \kappa E_M{}^0\ , \tag{2.39}$$

[8] In asymptotically flat spacetimes, it is customary to normalize the Killing vector such that $\xi^2 = -1$ at infinity. But this criterion cannot be applied in all cases, for example in AdS spacetimes. Therefore we will not impose any particular normalization.

[9] The idea behind this choice of vielbein is the following. First, we find E^ι so that it cointains all the contribution proportional to $\mathrm{d}\psi$. After that, we take a null vielbein of the form $\mathrm{d}s^2 = -2E^+E^- + (E^2)^2 + (E^3)^2 + \cdots + (E^\iota)^2$ with $E^+ \propto \mathrm{d}U$, and finally we convert it to a usual vielbein.

while $\mathcal{L}_\xi E_M{}^{a'} = \mathcal{L}_\xi E_M{}^{\iota} = 0$. Using (2.13), we see that $(\lambda^E_\xi)_{01} = -(\lambda^E_\xi)_{10} = -\kappa$ while the remaining components vanish; therefore we have

$$\mathrm{d}\lambda^E_\xi = 0 \ . \tag{2.40}$$

Regarding the remaining fields, we will consider $\mathcal{L}_\xi \Phi = \mathcal{L}_\xi B = 0$. This leads to the third good feature of the previous vielbein: the stationarity condition $\delta_\xi B = 0$ is simplified with (2.40) to the form $\delta_\xi B = \mathrm{d}\alpha_\xi \cong 0$. In this way, we will take $\alpha_\xi = 0$ in what follows. The reader should keep in mind that $\mathrm{d}\lambda^E_\xi = 0$ only holds in a neighborhood of the horizon covered by U, V, x^α. The knowledge of the fields in such neighborhood is the only necessary data to compute the entropy and the temperature, which are invariant under anomalous Lorentz transformations. Finally, regarding the U(1) isometry, we must have $\partial_\psi E_M{}^A = \partial_\psi B_{MN} = \partial_\psi \Phi = 0$ to perform the T-duality transformation, and we also require $G_{\psi\psi} \neq 0$ everywhere to prevent curvature singularities in the T-dual solution.

2.3.2 Properties of the corrected T-dual

The previous discussion shows that we can use the expression (2.31) with $S_{\alpha_\xi} = 0$ to compute the black hole entropy of a stationary black hole solution of the generalized BdR theory, provided it satisfies all the assumptions on the fields we have just presented. In particular, equations (2.12) are satisfied on-shell for a trivial α_ξ. If we T-dualize this solution with the corrected rules of the previous chapter, (1.49), before computing the entropy of the dual we must first check that the same conditions $\delta_\xi \hat{\Psi} = 0$ are met for the T-dual fields, namely that:

$$\delta_\xi \hat{E}_M{}^A = \delta_\xi \hat{B}_{MN} = \delta_\xi \hat{\Phi} = 0 \ . \tag{2.41}$$

These conditions will be shown to follow automatically from $\mathcal{L}_\xi \Omega^{(k)2}_{MN} = \mathcal{O}(a_\pm)$ and $\mathcal{L}_\xi \tilde{\Omega}^{(k)2}_{MN} = \mathcal{O}(a_\pm)$, so let us focus on these identities. Recall that tilde means here the leading order dual, which is obtained by means of the conventional Buscher rules. We begin by noting that, from (2.21):

$$\delta_\xi \Omega_A{}^B = \mathcal{L}_\xi \Omega_A{}^B + \mathrm{d}(\lambda^E_\xi)_A{}^B + \Omega_A{}^C (\lambda^E_\xi)_C{}^B - (\lambda^E_\xi)_A{}^C \Omega_C{}^B = 0 \ , \tag{2.42a}$$

$$\delta_\xi \Omega^{(k)B}_A = \mathcal{L}_\xi \Omega^{(k)B}_A + \Omega^{(k)C}_A (\lambda^E_\xi)_C{}^B - (\lambda^E_\xi)_A{}^C \Omega^{(k)B}_C + \mathcal{O}(a_\pm) = 0 \ , \tag{2.42b}$$

which vanish because $\delta_\xi E_M{}^A = 0$ and $\delta_\xi B = 0$ in the original solution. Since $\mathrm{d}\lambda^E_\xi = 0$, we can simplify the latter expression to obtain:

$$\mathcal{L}_\xi \Omega^{(k)B}_A + \mathrm{d}(\lambda^E_\xi)_A{}^B + \Omega^{(k)C}_A (\lambda^E_\xi)_C{}^B - (\lambda^E_\xi)_A{}^C \Omega^{(k)B}_C = \mathcal{O}(a_\pm) \ . \tag{2.43}$$

This is telling us that the leading order effect of the Lie derivative on $\Omega^{(k)B}_A$ is exactly a homogeneous Lorentz transformation with generator $-(\lambda^E_\xi)_A{}^B$. From the previous equation one easily arrives to:

$$\mathcal{L}_\xi \Omega^{(k)2}_{MN} = \mathcal{O}(a_\pm) \ . \tag{2.44}$$

Let us address now the T-dual configuration. Since we want to repeat the argument above, we show first that $\delta_\xi \tilde{E}_M{}^A = 0$ and $\mathcal{L}_\xi \tilde{B} = 0$. Indeed, under uncorrected Buscher rules (1.6), the only component of the vielbein with a non-trivial transformation is:

$$\tilde{E}^\iota = e^{-\sigma} \left(W_\mu \mathrm{d}x^\mu + \mathrm{d}\psi \right) \ . \tag{2.45}$$

Since $\mathcal{L}_\xi\sigma = 0$ and $\mathcal{L}_\xi W_\mu = -\mathcal{L}_\xi B_{\psi\mu} = 0$, it immediately follows that $\mathcal{L}_\xi \tilde{E}^\iota = 0$. The Lie derivative acts therefore on the Buscher-transformed vielbein in the same way it does on the original one, (2.39):

$$\mathcal{L}_\xi \tilde{E}_M{}^0 = \kappa \tilde{E}_M{}^1 \ , \qquad \mathcal{L}_\xi \tilde{E}_M{}^1 = \kappa \tilde{E}_M{}^0 \ , \tag{2.46}$$

while $\mathcal{L}_\xi \tilde{E}_M{}^{a'} = \mathcal{L}_\xi \tilde{E}_M{}^\iota = 0$. This means that $\delta_\xi \tilde{E}_M{}^A = \mathcal{L}_\xi \tilde{E}_M{}^A + \tilde{E}_M{}^B (\lambda_\xi^{\tilde{E}})_B{}^A = 0$, where the only independent non-vanishing component of $\lambda_\xi^{\tilde{E}}$ is $(\lambda_\xi^{\tilde{E}})_{01} = -\kappa$, and $\mathrm{d}(\lambda_\xi^{\tilde{E}})_A{}^B = 0$. Furthermore, $\mathcal{L}_\xi \tilde{B} = 0$ because of Buscher rules, (1.7). Taking $\tilde{\alpha}_\xi = 0$, we have fulfilled the condition $\delta_\xi \tilde{B} = 0$:

$$\delta_\xi \tilde{B} = \mathcal{L}_\xi \tilde{B} + \frac{1}{4}\left(a_+ \tilde{\Omega}_A^{(+)B} - a_- \tilde{\Omega}_A^{(-)B}\right) \wedge \mathrm{d}(\lambda_\xi^{\tilde{E}})_B{}^A + \mathrm{d}\tilde{\alpha}_\xi = 0 \ . \tag{2.47}$$

Therefore, we can repeat the reasoning applied before T-duality to conclude that $\mathcal{L}_\xi \tilde{\Omega}^{(k)2}_{MN} = \mathcal{O}(a_\pm)$. Armed with this result and (2.44), it is easy to see in the corrected rules (1.49) that $\mathcal{L}_\xi \hat{G}_{MN} = \mathcal{L}_\xi \hat{B}_{MN} = \mathcal{L}_\xi \hat{\Phi} = 0$. This is enough to ensure the conditions $\delta_\xi \hat{\Psi} = 0$ for all fields taking $\hat{\alpha}_\xi = 0$ and using $\mathrm{d}\lambda_\xi^{\hat{E}} = \mathcal{O}(a_\pm)$:

$$\delta_\xi \hat{E}_M{}^A = 0 \ , \tag{2.48a}$$

$$\delta_\xi \hat{B} = \mathcal{L}_\xi \hat{B} + \frac{1}{4}\left(a_+ \hat{\Omega}_A^{(+)B} - a_- \hat{\Omega}_A^{(-)B}\right) \wedge \mathrm{d}(\lambda_\xi^{\hat{E}})_B{}^A + \mathrm{d}\hat{\alpha}_\xi = 0 \ , \tag{2.48b}$$

$$\delta_\xi e^{-2\hat{\Phi}} = \delta_\xi \left(e^{-2\Phi}\sqrt{G/\hat{G}}\right) = 0 \ . \tag{2.48c}$$

The statement about the vielbein invariance follows from the definition of $\lambda_\xi^{\hat{E}}$ and the fact that $\mathcal{L}_\xi \hat{G}_{MN} = 0$, [85].

The object $e^{-2\Phi}\sqrt{G_{\mathcal{B}}}$, where $G_{\mathcal{B}}$ is the determinant of the induced metric in the bifurcation surface, will play a major role in the entropy invariance. This is expected from the form of the entropy, (2.31), since being the only part in that expression which is not multiplied by $\gamma_\pm$ we will need to know its transformation properties under the full T-duality corrected rules. For the remaining terms, leading order rules are enough, as they are multiplied by $\gamma_\pm$. To evaluate $e^{-2\hat{\Phi}}\sqrt{\hat{G}_{\mathcal{B}}}$, let us start with the following property:

$$G_{\psi\mu'}|_{\mathcal{B}} = B_{\psi\mu'}|_{\mathcal{B}} = 0 \ , \tag{2.49}$$

where we remind the reader that μ' can be either U or V. The metric component $G_{\psi\mu'}|_{\mathcal{B}}$ can be read from (2.37) at $U = V = 0$ (in fact in the whole horizon $V = 0$). $B_{\psi\mu'}|_{\mathcal{B}}$ is derived from $\mathcal{L}_\xi B_{MN} = 0$, evaluating the expansion of the Lie derivative at the bifurcation surface. From Buscher rules, it follows that:

$$\tilde{G}_{\psi\mu'}|_{\mathcal{B}} = \tilde{G}_{\alpha'\mu'}|_{\mathcal{B}} = 0 \ , \qquad \tilde{G}_{\mu'\nu'}|_{\mathcal{B}} = G_{\mu'\nu'}|_{\mathcal{B}} \ . \tag{2.50}$$

We can use these results in the expression of the corrected T-dual fields, (1.49). Furthermore, we can resort to $\mathcal{L}_\xi \Omega^{(k)2}_{MN} = \mathcal{O}(a_\pm)$ and $\mathcal{L}_\xi \tilde{\Omega}^{(k)2}_{MN} = \mathcal{O}(a_\pm)$ to make all $\Omega^{(k)2}_{MN}$ components appearing in the expressions of $\hat{G}_{\psi\mu'}|_{\mathcal{B}}$ and $\hat{G}_{\alpha'\mu'}|_{\mathcal{B}}$ vanish. Indeed, for any regular tensor T_{MN}:

$$\mathcal{L}_\xi T_{MN} = \mathcal{O}(a_\pm) \qquad \Rightarrow \qquad T_{\mu'\alpha'}|_{\mathcal{B}} = \mathcal{O}(a_\pm) = T_{\mu'\psi}|_{\mathcal{B}} \ . \tag{2.51}$$

Notice that $\tilde{\Omega}^{(k)}_{MA}{}^{B}$ is regular because $\tilde{E}_M{}^A$, $\tilde{E}_A{}^M$, and $\tilde{B}_{MN}$ are; see (1.8).[10] Thereby, the desired property follows:

$$\tilde{\Omega}^{(k)2}_{\mu'\alpha'}|_{\mathcal{B}} = \mathcal{O}(a_\pm) = \tilde{\Omega}^{(k)2}_{\mu'\psi}|_{\mathcal{B}} \ , \qquad \Omega^{(k)2}_{\mu'\alpha'}|_{\mathcal{B}} = \mathcal{O}(a_\pm) = \Omega^{(k)2}_{\mu'\psi}|_{\mathcal{B}} \ . \tag{2.52}$$

Substituting back in the corrected rules, we find that:

$$\hat{G}_{\psi\mu'}|_{\mathcal{B}} = \hat{G}_{\alpha'\mu'}|_{\mathcal{B}} = 0 \ , \qquad \hat{G}_{\mu'\nu'}|_{\mathcal{B}} = G_{\mu'\nu'}|_{\mathcal{B}} + \sum_{k=\pm} \frac{a_k}{4} \left(\tilde{\Omega}^{(k)2}_{\mu'\nu'} - \Omega^{(k)2}_{\mu'\nu'} \right) \Big|_{\mathcal{B}} \ . \tag{2.53}$$

The last step we need to perform is to show that the last two terms in $\hat{G}_{\mu'\nu'}|_{\mathcal{B}}$ cancel each other. To see that this is the case, we convert curved U, V indices to vielbein components $0, 1$; taking into account that $E_M{}^0|_{\mathcal{B}}$ and $E_M{}^1|_{\mathcal{B}}$ are non-vanishing only when $M = \mu'$, (2.38). A simple application of the leading order transformation rules, (1.48), then gives the desired result, $\tilde{\Omega}^{(k)2}_{\mu'\nu'} = \Omega^{(k)2}_{\mu'\nu'}$. Therefore:

$$\hat{G}_{UV}|_{\mathcal{B}} = G_{UV}|_{\mathcal{B}} \ , \qquad \hat{G}_{UU}|_{\mathcal{B}} = 0 \ , \qquad \hat{G}_{VV}|_{\mathcal{B}} = 0 \ . \tag{2.54}$$

The corrected dual metric has then a very simple block structure, which guarantees invariance under corrected T-duality of the components normal to the bifurcation surface $U = V = 0$:

$$\hat{G}_{MN}|_{\mathcal{B}} = \begin{pmatrix} 0 & G_{UV}|_{\mathcal{B}} & 0 \\ G_{UV}|_{\mathcal{B}} & 0 & 0 \\ 0 & 0 & \hat{G}_{\alpha\beta}|_{\mathcal{B}} \end{pmatrix} \ . \tag{2.55}$$

Notice that $\hat{G}_{UU}|_{\mathcal{B}} = \hat{G}_{VV}|_{\mathcal{B}} = \hat{G}_{\mu'\alpha}|_{\mathcal{B}} = 0$ also follow from $\mathcal{L}_\xi \hat{G}_{MN} = 0$. This convenient block structure allows to establish:

$$e^{-2\hat{\Phi}} \sqrt{\hat{G}_{\mathcal{B}}}\Big|_{\mathcal{B}} = e^{-2\hat{\Phi}} \frac{\sqrt{-\hat{G}}}{\sqrt{-\hat{G}_\perp}}\Bigg|_{\mathcal{B}} = e^{-2\Phi} \frac{\sqrt{-G}}{\sqrt{-G_\perp}}\Bigg|_{\mathcal{B}} = e^{-2\Phi} \sqrt{G_{\mathcal{B}}}\Big|_{\mathcal{B}} \ , \tag{2.56}$$

where in the second step we used that $e^{-2\Phi}\sqrt{-G}$ is invariant under corrected T-duality, and that $\hat{G}_\perp = -G^2_{UV}|_{\mathcal{B}} = G_\perp$ is the determinant of the metric restricted to the normal directions to $\mathcal{B}$, which is also invariant based on (2.55).

2.3.3 Entropy and temperature invariance

Based on the previous technical results, let us now address the question of the entropy and temperature invariance. Before proceeding to compute the entropy of the T-dual solution, it is necessary to show that we actually have a bifurcate Killing horizon after the corrected T-duality rules are applied to a black hole spacetime. A basic requirement is the regularity of the dual metric, which follows from $G_{\psi\psi} \neq 0$ and the non-singular $\Omega_{MA}{}^B$ and $\Omega^{(\pm)B}_{MA}$ before duality. Furthermore, in (1.48) we see that $\tilde{\Omega}^{(\pm)B}_{MA}$ must be regular. Then, we obtain a regular dual metric when we apply the corrected T-duality rules.

[10] The reader should keep in mind that we always assume $G_{\psi\psi} \neq 0$, as already mentioned. We also rely upon $e^{-2\tilde{\Phi}}\sqrt{-\tilde{G}} = e^{-2\Phi}\sqrt{-G}$. Using this expression, one can prove that the determinant satisfies $\det \tilde{E}_M{}^A = G^{-1}_{\psi\psi} \det E_M{}^A$, and then $\tilde{E}_A{}^M$ is regular.

In order to have a bifurcate Killing horizon one needs a Killing vector that is null on the horizon and vanishes on a codimension-2 surface. In fact, the same Killing field ξ of G_{MN} will be a Killing field of $\hat{G}_{MN}$; as shown in the previous discussion, all T-dual fields are invariant under the flow of ξ, $\mathcal{L}_\xi \hat{G}_{MN} = \mathcal{L}_\xi \hat{B}_{MN} = \mathcal{L}_\xi \hat{\Phi} = 0$. Furthermore, ξ^M still vanishes at $U = V = 0$. To show that it is null and orthogonal to the horizon we follow an argument similar to that of [86]. As $\hat{G}_{MN}$ is regular and $\xi^M|_{\mathcal{B}} = 0$, the scalars $\xi^M \xi^N \hat{G}_{MN}|_{\mathcal{B}} = \xi^M (\partial_\alpha)^N \hat{G}_{MN}|_{\mathcal{B}} = 0$. Moreover, $\mathcal{L}_\xi (\partial_\alpha)^M = 0$, so these scalars are invariant under the flow of ξ. From any point of $UV = 0$, one can get arbitrarily close to $\mathcal{B}$ through this flow. By continuity, the scalars $\xi^M \xi^N \hat{G}_{MN}$ and $\xi^M (\partial_\alpha)^N \hat{G}_{MN}$ also vanish for any point in $UV = 0$. There is a spacelike codimension-2 surface where ξ vanishes, namely $U = V = 0$. In the remaining points of $UV = 0$, there is a non-zero normal Killing vector ξ with respect to the metric $\hat{G}_{MN}$. Consequently, there is a bifurcate Killing horizon in $UV = 0$ after the duality [86, 89].

There is an aspect of the T-dual configuration which is not determined by the corrected T-duality rules; namely, the range of the dual coordinate. In the case of string theory, calculations using the path-integral of the underlying worldsheet description show that the ranges should be equal for a compact U(1) isometry, $\Delta\psi = \widehat{\Delta\psi} = 2\pi$ [96]. We asume this is the case for all values of $a_\pm$; otherwise, we would both spoil the entropy invariance for $a_\pm = 0$ already found in [86], and the invariance of the action under T-duality, as long as the Lagrangian itself is invariant.

Let us now present the expression for the entropy in terms of our vielbein. Before applying T-duality, the binormal to the bifurcation surface is given by $n = E^0 \wedge E^1|_{\mathcal{B}}$, (2.38). We have also argued that we can safely take $\alpha_\xi = 0$, so evaluating the entropy from the general expression (2.31), we obtain:

$$S = \frac{\pi}{\kappa_s^2} \int_{\mathcal{B}} \mathrm{d}^{D-2}x \, e^{-2\Phi} \sqrt{G_{\mathcal{B}}} \left[2 + 4\gamma_+ \left(R^{01}{}_{01} - \frac{3}{4} H^{A01} H_{A01} \right) - 4\gamma_- \Omega^{A01} H_{A01} \right] , \quad (2.57)$$

where $\mathrm{d}^{D-2}x \equiv \mathrm{d}\psi \, \mathrm{d}^{D-3}x$. After T-duality, the components of the binormal at the bifurcation surface are the same, due to the fact that the normal part of the metric is preserved, (2.55). In turn, when going to flat indices, this implies $\hat{n}_{AB}|_{\mathcal{B}} = n_{AB}|_{\mathcal{B}} + \mathcal{O}(a_\pm)$, because the leading order vielbein components E^0 and E^1 are invariant. The leading order part of $\hat{n}_{AB}|_{\mathcal{B}}$ is enough for our computation, because the binormals in the general entropy expression already appear multiplied by $\gamma_\pm$. Since $\hat{\alpha}_\xi = 0$, the integrand of the entropy after T-duality is given by the same expression (2.57), just placing a hat on each field.

The next step is to relate the integrands before and after duality. We have already done most of the work, beacause in (2.56) we showed that the leading order part of (2.57) is invariant on its own:

$$e^{-2\hat{\Phi}} \sqrt{\hat{G}_{\mathcal{B}}} \Big|_{\mathcal{B}} = e^{-2\Phi} \sqrt{G_{\mathcal{B}}} \Big|_{\mathcal{B}} . \quad (2.58)$$

There only remains to investigate how the expressions multiplied by $\gamma_\pm$ in (2.57) transform under T-duality. To the order we are working, we can use the leading order Buscher rules, and since our vielbein (2.38) and its leading order T-dual are of the class specified in (1.9), we can use the dimensionally-reduced T-duality rules (1.6): $\sigma \to -\sigma$ and $V_\mu \leftrightarrow W_\mu$. To do so, one has to perform first the dimensional reduction:

$$R^{01}{}_{01} - \frac{3}{4}H^{A01}H_{A01} = r^{01}{}_{01} - \frac{3}{4}h^{a01}h_{a01} - \frac{3}{4}\left(e^{2\sigma}V^{01}V_{01} + e^{-2\sigma}W^{01}W_{01}\right) \ , \tag{2.59a}$$

$$\Omega^{A01}H_{A01} = \omega^{a01}h_{a01} - \frac{1}{2}V^{01}W_{01} \ . \tag{2.59b}$$

The reduced Riemann tensor, field strength and Lorentz-connection $r^{01}{}_{01}$, h^{a01} and ω^{a01} are invariant up to $\mathcal{O}(a_\pm)$ terms. The leading order reduced rules change σ into $-\sigma$ and V into W, and as a consequence, both of the previous expressions are T-dual invariant to leading order. We have attained entropy invariance:

$$\hat{S} = S \ . \tag{2.60}$$

It is also possible to derive the T-dual invariance of the temperature, $T_H = \kappa/2\pi$, using the form of $\hat{G}_{MN}|_\mathcal{B}$, (2.55). The dual surface gravity $\hat{\kappa}$ can be easily computed in the bifurcation surface:

$$\hat{\kappa}\,\hat{n}_M{}^N|_\mathcal{B} = \hat{\nabla}_M\xi^N|_\mathcal{B} = \partial_M\xi^N|_\mathcal{B} = \kappa\, n_M{}^N|_\mathcal{B} \ . \tag{2.61}$$

The second equality follows from $\xi^M|_\mathcal{B} = 0$, and the latter is the consequence of $\nabla_M\xi^N|_\mathcal{B} = \kappa\, n_M{}^N|_\mathcal{B}$. Notice how $\partial_M\xi^N|_\mathcal{B}$ does not depend on the dual fields at all. From the coincident form before and after the duality of the normal part of the metric to the bifurcation surface, (2.55), it follows that $\hat{n}_M{}^N|_\mathcal{B} = n_M{}^N|_\mathcal{B}$. As a consequence, we obtain:

$$\hat{\kappa} = \kappa \ . \tag{2.62}$$

Therefore, we have established the T-dual invariance of the temperature.

2.4 Final discussion and conclusions

These last sections have been highly technical in content, so let us pause for a moment and take stock of results. The main conclusion is clear: spacetimes with a bifurcate Killing horizon in the generalized BdR theory are mapped under corrected T-duality to spacetimes which also have a bifurcate Killing horizon. In this process, both the temperature and entropy of the horizon are preserved, and this is true irrespective of the values of the perturbative parameters of the theory, $a_\pm$. This does not allow us to conclude that T-dual solutions are completely equivalent in terms of their physical properties yet – *e.g.*, we could ask what happens with other asymptotic charges –, but it certainly points towards this being the case. Total equivalence was certainly expected for string theory values of the parameters. The general result for all $a_\pm$ is however surprising: it points towards the possibility of having T-duality as a physical equivalence between radically different backgrounds in low-energy theories which are not directly related to string sigma models. Thus, it leads us to think of T-duality as a true equivalence of solutions to low-energy invariant actions more than just as a mere solution-generating technique.

In retrospect, this result may seem somehow expected from the fact that the corrected T-duality transformations in the BdR theory are a sequence of field redefinitions followed by the uncorrected Buscher rules and, finally, corresponding inverse field redefinitions –

this was the idea behind the construction of the corrected rules in the previous chapter. Each of those operations are expected to preserve surface gravity and entropy on their own. However, as we have shown in this chapter, an explicit check is not immediately trivial, and it requires powerful techniques to be done. Along the way, there are interesting conclusions that can be drawn, particularly the non-trivial interplay of anomalous Lorentz and gauge symmetry which allows to show the invariance of the entropy, (2.35).

3

The BTZ black hole/string example

This chapter intends to be a specific and down to earth example complementing the more abstract and general construction of the previous ones. It is always reassuring to check in an explicit case the validity of our general arguments. We will start from a black hole solution of the generalized BdR theory, generate by means of the corrected rules obtained in chapter 1 the corresponding T-dual (which also possesses a Killing horizon), and finally compute the entropy and temperature of both backgrounds, checking the precise matching.

The solution we will employ is based on the BTZ black hole, [97, 98]. This is a remarkable 3-dimensional black hole solution of General Relativity with negative cosmological constant, locally indistinguishable from pure AdS. As shown in [99], when supplemented with a trivial dilaton and a quadratic B-field, the BTZ black hole becomes a solution to the one-loop low-energy equations of motion of string theory, (1.1). Under leading order T-duality, this solution becomes the 3-dimensional black string obtained in [100]. This is particularly noticeable, because the BTZ black hole is asymptotically AdS, while the black string is asymptotically flat. The T-duality equivalence of both solutions suggests that strings in 3-dimensions are not sensitive to this asymptotic structure, and it demonstrates the extremely different properties a spacetime and its T-dual might have. Thus, this is a particularly suitable situation to test the entropy and temperature invariance under T-duality presented in the previous chapter.

In fact, this problem was studied to leading order in α' in [86]. There, after providing general arguments for the entropy and temperature invariance under leading order T-duality transformations, the BTZ black hole/string example was studied, explicitly showing the invariance. Our goal in the present chapter will be to perform a similar check in the generalized BdR theory, (1.31). Recall that this theory includes the two-loop corrections to the low-energy string effective action; but it goes beyond that, because for general values of the parameters $a_\pm$ we do not know of any sigma model that produces the action as the one governing its low-energy dynamics. We will generalize the BTZ solution to this corrected theory, we will compute its T-dual under the corresponding corrected rules and, finally, we will explicitly check that the entropy and temperature invariance is satisfied, as we know it must based on the general arguments of the previous chapter. Along the way, we will also provide the black string solution including its first perturbative corrections, since it is the T-dual of the generalized BTZ solution.

3.1 The BTZ solution

The BTZ solution to the one-loop low-energy theory presented in [99] has the following form:[1]

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{N^2} + r^2\left(\mathrm{d}\psi + V_t\,\mathrm{d}t\right)^2 \ , \tag{3.1a}$$

$$e^{-2\Phi} = 1 \ , \tag{3.1b}$$

$$B = \frac{r_+^2 - r^2}{\ell}\mathrm{d}t \wedge \mathrm{d}\psi \ , \tag{3.1c}$$

with the lapse given by the standard BTZ form,

$$N^2 \equiv \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} \ , \tag{3.2}$$

and where

$$V_t \equiv \frac{r_+ r_-}{\ell}\left(\frac{1}{r_+^2} - \frac{1}{r^2}\right) . \tag{3.3}$$

Notice that, in these expressions, r_+ and r_- are two integration constants, $r_+ \geq r_-$, better expressed in terms of the ($a_\pm$-uncorrected) black hole mass and angular momentum computed from (1.2):

$$M = \frac{r_+^2 + r_-^2}{\ell^2} \ , \qquad J = \frac{2r_+ r_-}{\ell} . \tag{3.4}$$

This solution, being locally AdS_3, is valid to all orders in α' in low-energy string theory, since it can be obtained via a Wess-Zumino-Witten model for string propagation on a Lie group, [99]. Recall that in the $a_\pm$-corrected generalized BdR theory, (1.31), we are outside the realm of string theory, so in principle we are not guaranteed that this is still a solution. But the fact that for some particular values of $a_\pm$ (those corresponding to string theory, (1.35)) it must be so leads us to expect its validity for all $a_\pm$. This is indeed the case, but to check it we have to first select a vielbein, since the torsionful Chern-Simons forms appear in the equations of motion (1.37). The obvious choice does the job:

$$E^0 = N\mathrm{d}t \ , \qquad E^1 = \frac{\mathrm{d}r}{N} \ , \qquad E^2 = r\left(\mathrm{d}\psi + V_t\,\mathrm{d}t\right) \ . \tag{3.5}$$

Thus, (3.1) with this vielbein choice is a solution to the equations of motion (1.37) (with $\Lambda = -2/\ell^2$). In fact, it is so with trivial torsionful Riemann tensors and Chern-Simons forms, $R^{(\pm)}_{MNAB} = \Theta^{(\pm)}_{MNR} = 0$, so that the equations of motion actually reduce to the leading order ones.

The vielbein choice (3.5), however, is problematic. This can be seen in various ways, probably the most rigorous one being changing to Kruskal-like coordinates and checking that some of its components diverge at the bifurcation surface of the external horizon, located at $r = r_+$, where the lapse function (3.2) vanishes. This can already be anticipated from the form of (3.5): since r should be a good coordinate covering the future

[1] This is a solution to the leading order equations of motion if we also include a cosmological constant, which we did not do in (1.1). It is easier to see the form of these equations if we set $a_\pm = 0$ and $\Lambda = -2/\ell^2$ in (1.37).

exterior horizon, E^1 will diverge there as a consequence of $N(r_+) = 0$. There is a faster way to check the irregular character of our vielbein, which is even more illuminating for our purposes, because it connects nicely with the regularity condition for the fields we imposed in the previous chapter when computing the entropy. First of all, the Killing field generating the horizon $r = r_+$ is $\xi = \partial_t$. This is a consequence of the fact that both functions N and N^ψ vanish at r_+, see (3.2) and (3.3), so that ξ is null there.[2] Now, the vielbein (3.5) is invariant under the flow of this ξ, therefore:

$$0 = \mathcal{L}_\xi E^A = \mathrm{d}i_\xi E^A + i_\xi \mathrm{d}E^A = \mathrm{d}i_\xi E^A - \Omega_B{}^A i_\xi E^B + E^B i_\xi \Omega_B{}^A \ , \tag{3.6}$$

where we used the definition of the spin connection, $\nabla_M E_N{}^A = \Omega_{MB}{}^A E_N{}^B$, which implies $\mathrm{d}E^A = \Omega_B{}^A \wedge E^B$. The first two terms of the previous equation can be further simplified using again the definition of the spin connection, and after some algebraic manipulations we arrive at:

$$i_\xi \Omega_B{}^A = \xi^M \Omega_{MB}{}^A = -E_N{}^A E_B{}^M \nabla_M \xi^N \ . \tag{3.7}$$

If we evaluate this at the bifurcation surface, $\mathcal{B}$, the right hand side gives $-\kappa n_B{}^A$, with n_{MN} the binormal to $\mathcal{B}$ and κ the surface gravity. But the left hand side is a contraction with the Killing field ξ, which vanishes at $\mathcal{B}$. The only possible way the contraction can be non-vanishing and equal to the right hand side is by means of a singular spin connection at $\mathcal{B}$. Thus, if we work with a vielbein invariant under the flow of ξ – our only assumption to arrive to (3.7) –, we necessarily have a singular spin connection. In this situation, our derivation of the entropy formula in the previous chapter is not valid, so we must choose a different vielbein which respects the regularity condition.

The solution turns out to be obtained after a simple local Lorentz boost in the radial direction.[3] From now on, we take our vielbein to be:

$$E^0 = \cosh(\kappa t)\, N \mathrm{d}t + \sinh(\kappa t)\, \frac{\mathrm{d}r}{N} \ , \quad E^1 = \sinh(\kappa t)\, N \mathrm{d}t + \cosh(\kappa t)\, \frac{\mathrm{d}r}{N} \ , \tag{3.8}$$

with E^2 given by the same expression in (3.5). This vielbein is not invariant under the flow of ξ, but it can be shown to produce a regular spin connection, satisfying $\xi^M \Omega_{MA}{}^B|_{\mathcal{B}} = 0$. We must face now a different problem, though. Our action including $a_\pm$ corrections is not locally Lorentz invariant unless we take into account the anomalous transformation of the B-field, (1.36). So, if we plug the Chern-Simons form obtained from (3.8) together with the fields in (3.1) in the equations of motion of the generalized BdR theory, (1.37), they are not satisfied. We must find the transformation of the B-field which compensates the change of vielbein from the irregular one, (3.5), to the regular one, (3.8). The fastest way to do this is by noticing that the fields transforming non-trivially under local Lorentz transformations appear in the equations of motion only through the combination

$$H'_{MNR} = H_{MNR} - \frac{3}{2}\left(a_-\, \Theta^{(-)}_{MNR} - a_+\, \Theta^{(+)}_{MNR}\right) \ . \tag{3.9}$$

Therefore, the form of the $\Theta^{(\pm)}_{MNR}$ terms for the new vielbein (3.8) gives us the change of the H_{MNR} field needed to keep H'_{MNR} invariant (recall that $\Theta^{(\pm)}_{MNR}$ were trivial for

[2] We will show this more explicitly below, after introducing ingoing Eddington-Finkelstein coordinates covering the future exterior horizon.

[3] This can be motivated by going to Kurskal-like coordinates.

the irregular vielbein). This, in turn, guarantees we have a solution to the equations of motion. Once we have H_{MNR}, this defines the B-field up to a gauge transformation, and this last freedom can be used to guarantee regularity at the bifurcation surface, which means $\xi^M B_{MN}|_{\mathcal{B}} = B_{tN}|_{\mathcal{B}} = 0$.[4] All in all, the regular B-field which provides a solution to the generalized BdR equations of motion together with the vielbein (3.8) and the trivial dilaton is:

$$B = \left[\frac{r_+^2 - r^2}{\ell} + 2\kappa\frac{r_+ - r}{\ell}\left(\gamma_+ - \frac{\gamma_- r_-}{r}\right)\right] \mathrm{d}t \wedge \mathrm{d}\psi \ , \tag{3.10}$$

where we recall the relation between $a_\pm$ and $\gamma_\pm$ parameters, (1.55):

$$\gamma_\pm \equiv \mp\frac{a_- \pm a_+}{4} \ . \tag{3.11}$$

and the value of the surface gravity κ in terms of the parameters of the solution will be computed in the following discussion.

3.1.1 Thermodynamics

Let us now review the computation of the thermodynamic quantities associated with the BTZ solution. Most of this discussion will carry over to the corrected T-dual, so we will provide some level of detail here in order to be more sketchy later. The temperature of the black hole is proportional to the surface gravity κ, which is defined as $\xi^N \nabla_N \xi^M|_{\mathcal{N}} = \kappa\xi^M|_{\mathcal{N}}$. $\mathcal{N}$ here is any point along the horizon. Using the fact that ξ is a Killing vector, so that $\nabla_{(M}\xi_{N)} = 0$, we can rewrite the previous expression as $\nabla_M \xi^2|_{\mathcal{N}} = -2\kappa\xi_M|_{\mathcal{N}}$. Since $\xi|_{\mathcal{B}} = 0$, we have to evaluate the previous expression away from the bifurcation surface in order to obtain the surface gravity. For this we need a set of coordinates covering the future horizon; this is done as in the archetypical Schwarzschild case by introducing ingoing Eddington-Finkelstein coordinates:

$$\mathrm{d}v \equiv \mathrm{d}t + \frac{\mathrm{d}r}{N^2} \ , \qquad \mathrm{d}\tilde{\psi} = \mathrm{d}\psi - \frac{V_t}{N^2}\mathrm{d}r \ , \tag{3.12}$$

so that the metric becomes:

$$\mathrm{d}s^2 = -N^2\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\left(\mathrm{d}\tilde{\psi} + V_t\,\mathrm{d}v\right)^2 \ . \tag{3.13}$$

This metric is well-behaved at $r = r_+$, so we can extend past this value towards the interior. Incidentally, this change of coordinates proves also that $r = r_+$ is an horizon (in the same way one does in the Schwarzschild black hole, the key ingredient is that it is the first zero of $G_{vv} = -N^2$ one encounters from the outside) and its normal one-form is $\mathrm{d}r$, which is null at $r = r_+$ since

$$G^{MN}(\mathrm{d}r)_M(\mathrm{d}r)_N = N^2 + r^2(V_t)^2 \ , \tag{3.14}$$

vanishes at the horizon. Written as a vector, since both N^2 and V_t vanish at $r = r_+$, $(\mathrm{d}r)^M|_{\mathcal{N}} = (\partial_v)^M|_{\mathcal{N}}$. This justifies our previous claim that the generator of the horizon is

[4] Regularity at the bifurcation surface plus invariance under the flow of ξ, which generates the horizon, imply by continuity $B_{tN}(r = r_+) = 0$ in the whole horizon. This is the condition we impose.

$\xi = \partial_t$ in the original coordinates: going back to the change (3.12), we see that $\partial_v = \partial_t$. Finally, we can compute the surface gravity:

$$\nabla_M \xi^2\big|_{\mathcal{N}} = \partial_r \left[-N^2 + r^2 (V_t)^2\right] (\mathrm{d}r)_M\big|_{\mathcal{N}} = -\partial_r N^2 \, \xi_M\big|_{\mathcal{N}} = -2\frac{r_+^2 - r_-^2}{\ell^2 r_+} \, \xi_M\big|_{\mathcal{N}} \, , \tag{3.15}$$

where in the second equality we used $V_t\,|_{\mathcal{N}} = 0$, and regularity at the horizon. Thus, we read:

$$\kappa = \frac{r_+^2 - r_-^2}{\ell^2 r_+} \equiv \kappa_{\mathrm{BTZ}} \, . \tag{3.16}$$

The entropy of the BTZ solution in the generalized BdR theory is computed from the expression obtained in the previous chapter, (2.31). However, before carelessly substituting the values of the different tensors appearing there for the BTZ solution, we must check that the assumptions employed in its derivation are satisfied. There are two essential ingredients. One is regularity: all fields must be regular at $\mathcal{B}$, in such a way that any expression involving contractions with ξ^M in $\mathcal{B}$ gives zero. This is indeed satisfied: that was the whole point of changing the vielbein to the form (3.8), and the B-field was chosen in a gauge so as to ensure this, (3.10). The second important point is that the on-shell variation $\delta_\xi \Psi$ must vanish for all fields. Recall that, as indicated in (2.12), these conditions are:

$$\delta_\xi \Phi = \mathcal{L}_\xi \Phi \cong 0 \, , \tag{3.17a}$$

$$\delta_\xi E^A = \mathcal{L}_\xi E^A + E^B (\lambda_\xi^E)_B{}^A \cong 0 \, , \tag{3.17b}$$

$$\delta_\xi B = \mathcal{L}_\xi B - \frac{a_-}{4} \mathrm{d}(\lambda_\xi^E)_A{}^B \wedge \Omega_B^{(-)A} + \frac{a_+}{4} \mathrm{d}(\lambda_\xi^E)_A{}^B \wedge \Omega_B^{(+)A} + \mathrm{d}\alpha_\xi \cong 0 \, . \tag{3.17c}$$

The dilaton and the B-field are invariant under the Killing flow. The vielbein is not, but it satisfies the simple relations:

$$\mathcal{L}_\xi E^0 = \kappa E^1 \, , \qquad \mathcal{L}_\xi E^1 = \kappa E^0 \, , \qquad \mathcal{L}_\xi E^2 = 0 \, . \tag{3.18}$$

This implies, using the definition (2.13), that $\delta_\xi E^A \cong 0$ is satisfied with $(\lambda_\xi^E)^{01} = \kappa = -(\lambda_\xi^E)^{10}$, and the remaining components vanish. In particular, being κ constant, $\mathrm{d}\lambda_\xi^E = 0$, and the condition $\delta_\xi B \cong 0$ is satisfied with $\alpha_\xi = 0$. Thus, all conditions (3.17) are satisfied for the BTZ solution, and we can write the entropy as:

$$S = \frac{2\pi}{\kappa_s^2} \int_{\mathcal{B}} \mathrm{d}\psi \, e^{-2\Phi} \sqrt{G_{\mathcal{B}}} \left[1 + 2\gamma_+ \left(R^{01}{}_{01} - \frac{3}{4} H^{A01} H_{A01}\right) - 2\gamma_- \Omega^{A01} H_{A01}\right] \, , \tag{3.19}$$

where we have employed the fact that the bifurcation surface can be reached from the (t, r, ψ) patch in the limit $r \to r_+$ and t finite, so that, being a surface at constant t and r, the binormal is $n = \mathrm{d}t \wedge \mathrm{d}r = E^0 \wedge E^1$. Notice that in this case the bifurcation surface is just a curve parametrized by ψ. For the BTZ solution the induced determinant is $G_{\mathcal{B}} = r^2$, and evaluating also the remaining tensors appearing in the previous expression:

$$S = \frac{4\pi^2}{\kappa_s^2} r_+ + \frac{16\pi^2}{\kappa_s^2 \ell^2} (\gamma_+ r_+ + \gamma_- r_-) \equiv S_{BTZ} \, , \tag{3.20}$$

where we take 2π as the range of the coordinate ψ along the U(1) symmetry. Notice that for $\gamma_- \neq 0$ the entropy depends on the inner horizon radius r_-. This is standard in theories with broken parity, such as Topologically Massive Gravity [101] or Mielke-Baekler's gravity [102, 103]. For the heterotic string, $\gamma_+ = -\gamma_-$, and the entropy correction depends on the difference $r_+ - r_-$.

3.2 The T-dual corrected black string

Let us now apply the corrected T-duality rules of chapter 1, (1.49), to the previous BTZ solution. Recall that, for that rules to be valid, we must choose a vielbein before duality adapted to the U(1) isometry, in the form (1.9). Fortunately, the choice (3.8) already satisfies this criterion: E^0 and E^1 constitute a vielbein for the reduced metric in the (t, r) subspace, while E^2 is the only vielbein component with a term proportional to the dualizing coordinate one-form, $\mathrm{d}\psi$. Direct application of the corrected rules with this vielbein choice produces, starting from the regular solution with the B-field given by (3.10):

$$\mathrm{d}\hat{s}^2 = -N^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{N^2} + e^{-2\sigma}\left(\mathrm{d}\chi + \hat{V}_t\ \mathrm{d}t\right)^2 \ , \tag{3.21a}$$

$$e^{-2\hat{\Phi}} = r^2\left(1+\Delta_+\right) \ , \tag{3.21b}$$

$$\begin{aligned}\hat{B} = \frac{r_+r_-}{\ell r^2}\left(\frac{r}{r_+}-1\right)\Bigg[\left(\frac{r}{r_+}+1\right) &- \frac{2\gamma_+}{\ell^2}\frac{r_+^2+2rr_-+r_-^2}{r^2} \\ &+\frac{2\gamma_-}{\ell^2}\left(\frac{r_+^2-r_-^2}{r_+r_-}-\frac{2r_-}{r}-\frac{2r_+r_-}{r^2}\right)\Bigg]\mathrm{d}t\wedge\mathrm{d}\chi \ ,\end{aligned} \tag{3.21c}$$

where N^2 is the same lapse function as in (3.2), we use χ for the coordinate dual to ψ, and

$$\hat{V}_t = \frac{r_+^2-r^2}{\ell}\left(1+\frac{r_-}{r_+}\Delta_-\right) \ , \qquad e^{-\sigma} = \frac{1-\Delta_+}{r} \ . \tag{3.22}$$

with the following definitions:

$$\Delta_\pm = \frac{2}{\ell^2r^2}\left[\gamma_\pm\left(r_+^2+r_-^2\right)+2\gamma_\mp r_+r_-\right] \ . \tag{3.23}$$

Notice that the previous expression for the metric contains terms quadratic in $\gamma_\pm$. These must be discarded to the order we are working, their presence is just a convenient way to express the first order $\gamma_\pm$-corrected vielbein we are choosing, which is of the same form as (3.8). This solution represents a $\gamma_\pm$-corrected black string, as we can demonstrate through a clever rewriting. First, we extend the leading order coordinate transformation found in [86] to include $\gamma_\pm$ corrections in the following way:

$$t = \ell(r_+^2-r_-^2)^{-1/2}\left(1-\frac{2}{\ell^2}\left(\gamma_+-\frac{r_-}{r_+}\gamma_-\right)\right)(T+X) \ , \tag{3.24a}$$

$$\chi = -(r_+^2-r_-^2)^{1/2}\left[\left(1+\frac{2}{\ell^2}\left(\gamma_++\frac{r_-}{r_+}\gamma_-\right)\right)X+\frac{4\gamma_-r_-}{\ell^2r_+}T\right] \ , \tag{3.24b}$$

and $r^2 = \ell\rho$, thereby $r_\pm^2 = \ell\rho_\pm$. If we now further rewrite everything in terms of the conserved quantities of the leading order solution, $\mathcal{M} = \rho_+$ and $\mathcal{Q} = -\sqrt{\rho_+\rho_-}$:

$$
\begin{aligned}
\mathrm{d}\hat{s}^2 =& -\left(1-\frac{\mathcal{M}}{\rho}\right)\left[1-\frac{4\mathcal{M}}{\ell^2\rho}\left(\gamma_+ - \frac{\mathcal{Q}}{\mathcal{M}}\gamma_-\right)\right]\mathrm{d}T^2 \\
&- \frac{4\left(1-\mathcal{Q}^2/\mathcal{M}^2\right)\gamma_-\mathcal{Q}}{\ell^2\rho}\left(1+\frac{\mathcal{M}}{\rho}\right)\mathrm{d}T\,\mathrm{d}X \\
&+ \left(1-\frac{\mathcal{Q}^2}{\mathcal{M}\rho}\right)\left[1+\frac{4\mathcal{Q}}{\ell^2\rho}\left(\gamma_- - \frac{\mathcal{Q}}{\mathcal{M}}\gamma_+\right)\right]\mathrm{d}X^2 \\
&+ \left(1-\frac{\mathcal{M}}{\rho}\right)^{-1}\left(1-\frac{\mathcal{Q}^2}{\mathcal{M}\rho}\right)^{-1}\frac{\ell^2\mathrm{d}\rho^2}{4\rho^2} \qquad (3.25a) \\
e^{-2\hat{\Phi}} =& \ell\rho + \frac{2}{\ell}\left[\left(1+\frac{\mathcal{Q}^2}{\mathcal{M}^2}\right)\gamma_+\mathcal{M} - 2\gamma_-\mathcal{Q}\right] \ , \qquad (3.25b) \\
\hat{B} =& \frac{\mathcal{Q}}{\rho}\left[\left(\frac{\rho}{\mathcal{M}}-1\right) + \frac{2}{\ell^2}\gamma_+\left(\frac{\mathcal{M}^2+\mathcal{Q}^2}{\mathcal{M}\rho} + \frac{\mathcal{M}^2-\mathcal{Q}^2}{\mathcal{M}^{3/2}\rho^{1/2}} - 2\right)\right. \\
&\left. + \frac{2}{\ell^2}\gamma_-\left(\frac{\mathcal{M}^2+\mathcal{Q}^2}{\mathcal{M}\mathcal{Q}} - \frac{\mathcal{M}^2-\mathcal{Q}^2}{\mathcal{M}^{3/2}\mathcal{Q}}\rho^{1/2} - \frac{2\mathcal{Q}}{\rho}\right)\right]\mathrm{d}T\wedge\mathrm{d}X \ . \qquad (3.25c)
\end{aligned}
$$

Notice that the angular velocity at the horizon is non-vanishing in spite of the fact that $G_{TX} \to 0$ at infinity. In the case of bosonic string theory, $\gamma_- = 0$ and $\gamma_+ = \alpha'/2$, the metric becomes diagonal:

$$
\begin{aligned}
\mathrm{d}\hat{s}^2 =& -\left(1-\frac{\mathcal{M}}{\rho}\right)\left(1-\frac{2\mathcal{M}}{\rho}\frac{\alpha'}{\ell^2}\right)\mathrm{d}T^2 + \left(1-\frac{\mathcal{Q}^2}{\mathcal{M}\rho}\right)\left(1-\frac{2\mathcal{Q}^2}{\mathcal{M}\rho}\frac{\alpha'}{\ell^2}\right)\mathrm{d}X^2 \\
&+ \left(1-\frac{\mathcal{M}}{\rho}\right)^{-1}\left(1-\frac{\mathcal{Q}^2}{\mathcal{M}\rho}\right)^{-1}\frac{\ell^2\mathrm{d}\rho^2}{4\rho^2}, \qquad (3.26a) \\
e^{-2\hat{\Phi}} =& \ell\left[\rho + \left(1+\frac{\mathcal{Q}^2}{\mathcal{M}^2}\right)\mathcal{M}\frac{\alpha'}{\ell^2}\right] \ , \qquad (3.26b) \\
\hat{B} =& \frac{\mathcal{Q}}{\rho}\left[\left(\frac{\rho}{\mathcal{M}}-1\right) + \left(\frac{\mathcal{M}^2+\mathcal{Q}^2}{\mathcal{M}\rho} + \frac{\mathcal{M}^2-\mathcal{Q}^2}{\mathcal{M}^{3/2}\rho^{1/2}} - 2\right)\frac{\alpha'}{\ell^2}\right]\mathrm{d}T\wedge\mathrm{d}X \ . \qquad (3.26c)
\end{aligned}
$$

This is the leading (bosonic) stringy correction to the black string found in [100]:

$$
\mathrm{d}s^2 = -\left(1-\frac{\rho_+}{\rho}\right)\mathrm{d}T^2 + \left(1-\frac{\rho_-}{\rho}\right)\mathrm{d}X^2 + \left(1-\frac{\rho_-}{\rho}\right)^{-1}\left(1-\frac{\rho_+}{\rho}\right)^{-1}\frac{\ell^2\mathrm{d}\rho^2}{4\rho^2} \ . \qquad (3.27)
$$

Notice finally that we can study the asymptotic behavior of the corrected solution, (3.25), when $\rho \to \infty$:

$$
\mathrm{d}\hat{s}^2 \underset{\rho\to\infty}{\sim} -\mathrm{d}T^2 + \mathrm{d}X^2 + \frac{\ell^2 d\rho^2}{4\rho^2} \ , \qquad (3.28)
$$

with a linear dilaton, $e^{-2\hat{\Phi}} \sim \ell\rho$, and a pure gauge Kalb-Ramond two-form, $\hat{B}_{TX} = \mathcal{Q}/\mathcal{M}$. The line element is that of flat space, and the scalar curvature at leading order in $1/\rho$ reads:

$$
R = \frac{4}{\ell^2\rho}\left[\frac{\mathcal{M}^2+\mathcal{Q}^2}{\mathcal{M}}\left(1+\frac{4\gamma_+}{\ell^2}\right) - 2\mathcal{Q}\frac{\gamma_-}{\ell^2}\right] \underset{\rho\to\infty}{\sim} \mathcal{O}(\rho^{-1}) \ , \qquad (3.29)
$$

which is the expected fall-off of an asymptotically flat metric in three dimensions, [104]. This validates our previous claim that the T-dual of the BTZ (which is asymptotically AdS) is asymptotically flat.

3.2.1 Thermodynamics

We have already done most of the work needed to compute the thermodynamic quantities of the T-dual solution. This is due to the fact that the black string, in the form (3.21), is formally very similar to the original BTZ, (3.1). The computation of the temperature is in fact identical. Since $\hat{V}_t\,(r = r_+) = 0$ and the lapse function N is invariant under corrected T-duality, we can repeat the steps (3.15) to obtain exactly the same result:

$$\hat{\kappa} = \frac{r_+^2 - r_-^2}{\ell^2 r_+} \equiv \kappa_{\text{BTZ}}\ . \tag{3.30}$$

The entropy computation is also very similar. First of all, both the regularity conditions and the on-shell vanishing of the variations (3.17) are satisfied for the T-dual solution. Thus, the entropy is formally given by the same expression (3.19), just writing hats over all fields and integrating over the dual isometric coordinate χ instead of ψ. The only remaining step is to compute the different tensors appearing in that expression for the background (3.21). Once this is done, we find that the result precisely matches the previous one:

$$\hat{S} = \frac{4\pi^2}{\kappa_s^2} r_+ + \frac{16\pi^2}{\kappa_s^2 \ell^2}\left(\gamma_+ r_+ + \gamma_- r_-\right) \equiv S_{BTZ}\ , \tag{3.31}$$

thus showing explicitly the invariance of the horizon temperature and entropy in this example.

Part II

Holographic entanglement entropy in higher-curvature gravity

4

Perturbative holographic entanglement entropy in higher-curvature gravity

The AdS/CFT correspondence [21–23] is, up until now, the only concrete realization we possess of the holographic principle [18, 19]. This is an idea rooted in the area law behaviour of black hole entropy which heuristically states that, in quantum gravity, the number of degrees of freedom contained in a certain region of space is not proportional to its volume (as it is in a local quantum field theory), but to its area instead. The nice review [20] formulates this proposal with a greater level of precision, but the main idea is simple to state: we should not expect to describe a theory of quantum D-dimensional spacetime as a conventional D-dimensional quantum field theory, but as some kind of theory in one dimension less instead. The AdS/CFT correspondence makes this broad proposal somewhat more precise for a certain kind of spacetimes (those with AdS asymptotics) providing also an explicit dual description in one dimension less: a certain kind of conformal field theory. Given the fact that this holographic behaviour is one of the few hints we have towards the properties of a truly quantum gravitational theory, it is difficult to overstate the relevance of the AdS/CFT correspondence, as the huge amount of research it triggered in the last quarter of century demonstrates.

We will not discuss the AdS/CFT proposal in full detail – we have devoted some space to a general overview in the introduction, and we refer the reader to one of the many reviews present in the literature for further information [24–26, 28] –, but only sketch the key ideas needed to contextualize our work presented in this and the next chapter. The basic ingredient behind most constructions in AdS/CFT is the identification of the bulk partition function with prescribed boundary conditions with the generating functional of the boundary CFT. Recall that we take the bulk to be a $(d+1)$-dimensional asymptotically AdS spacetime, with possibly some fields living on it. This spacetime has a conformal boundary at spatial infinity in which we must impose boundary conditions for the fields. The CFT lives in this asymptotic boundary, and the boundary values of the fields act as sources of the dual field theory operators:

$$e^{-\mathcal{I}_{E,\text{bulk}}[\phi_0]} = \left\langle \exp\left(\int_{\partial\text{AdS}} \phi_0(x)\mathcal{O}(x)\right) \right\rangle_{\text{CFT}} , \tag{4.1}$$

where we have computed the bulk partition function in the semiclassical approximation, and $\mathcal{O}$ is the dual field theory operator to the bulk field ϕ.[1] Notice that in the previous expression, ϕ denotes a general bulk field (not necessarily a scalar), and the left hand side involves computing the on-shell bulk action with boundary conditions $\phi(z,x) \sim z^{\Delta-d}\phi_0(x)$ as we approach the AdS asymptotic boundary at $z=0$ (z is a Poincaré coordinate penetrating into the bulk, and Δ the dimension of the operator $\mathcal{O}$). Notice also that we work in Euclidean signature; we will always explicitly write subscript E in the action or Lagrangian when we do so. Equation (4.1) is a highly non-trivial result, as it implies that we can compute CFT quantities such as correlation functions using only bulk techniques, *i.e.*, computations in classical (super)gravity. Naturally, the CFT must satisfy some properties for it to be valid. This was discussed in the introduction, but essentially we expect the field theory to be strongly coupled and posses a large number of degrees of freedom.

The identification of CFT quantities with the corresponding bulk ones gives rise to the so called holographic dictionary of AdS/CFT. Among many entries of this dictionary, one of particular interest is that proposed by Ryu and Takayanagi for computing the entanglement entropy of a spatial region of the CFT [32, 105]. This proposal states that, when the bulk theory is Einstein gravity (with possibly additional matter fields), the entanglement entropy of a region A in the boundary CFT is given by the area of an associated bulk codimension-2 surface Γ_A as:

$$S^{\mathrm{E}}_{\mathrm{EE}}(A) = \frac{\mathrm{Area}(\Gamma_A)}{4G_N}\,, \tag{4.2}$$

where superscript E emphasizes here that the bulk theory is Einstein gravity. The bulk surface Γ_A is defined to be the minimal area one among all those homologous to the region A in the boundary (it has to end in ∂A if this is not empty). We will call it the RT or Ryu-Takayanagi surface, a sketch of this surface and its relation to the boundary region can be found in Figure 4.1. Notice that entanglement entropy in a local quantum field theory is a non-local quantity, extremely difficult to compute in general situations. This makes equation (4.2) even more astonishing: AdS/CFT has turned an intrinsically quantum and non-local quantity into a purely geometric notion. It is perhaps this deep connection between quantum physics and geometry what explains the huge impact the Ryu-Takayanagi proposal had on the research in holography and AdS/CFT. It would be difficult to go through all the fundamental developments the holographic entanglement entropy program has triggered, but let us just mention that even such a long-standing problem as the discussion on the unitarity of black hole evaporation has received new perspectives recently [106, 107] which rely in an essential way on (4.2) and on the subsequent developments that allowed to include bulk quantum corrections, [108, 109]. It has even been suggested that geometry and gravitational dynamics in the bulk emerge as a consequence of quantum entanglement in the dual theory, [110, 111], thus indicating a deep connection between gravity and entanglement.

As already mentioned, (4.2) is only valid when the bulk gravitational theory is Einstein gravity with possibly some matter fields. Higher-curvature corrections can appear in the bulk action of (4.1) as stringy corrections, and they are particularly interesting in the AdS/CFT context because they define holographic field theory duals which are

[1] See the Introduction for an extended discussion about the identity of bulk and boundary partition functions, and its form before taking the saddle point approximation.

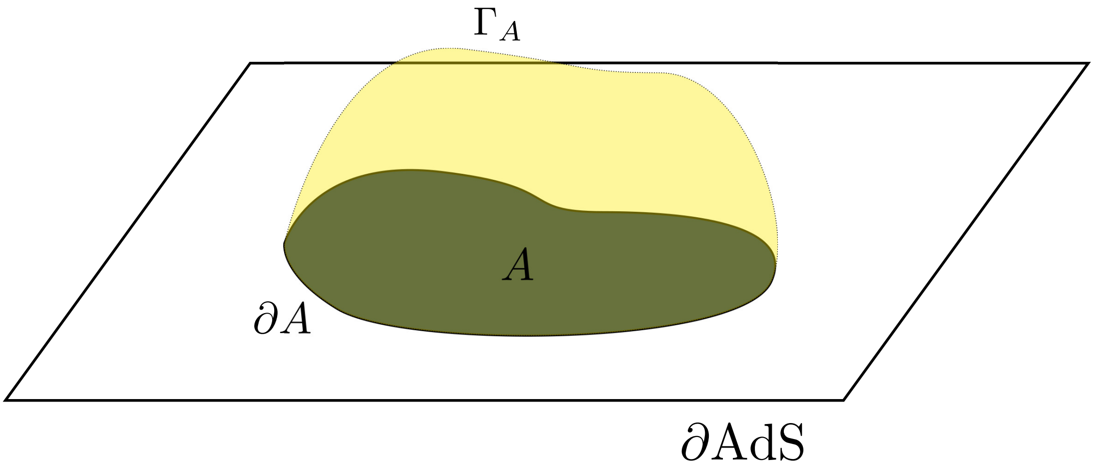

Figure 4.1: Given an arbitrary region A of the CFT, we identify it with the corresponding region in the AdS conformal boundary ∂AdS. The RT surface Γ_A is then the minimal area one among all those penetrating into the bulk and ending in ∂A. The picture represents a fixed time slice (the RT prescription is only valid in the absence of time dependence).

inequivalent to the Einstein gravity one. One natural question is then whether we can extend the RT proposal to higher-curvature gravity. Following the methods of [34], in the works [52, 53] the answer has been given in the affirmative. Our work in this chapter will be to provide a general overview of how this is achieved, writing an explicit expression to holographically compute entanglement entropy of boundary regions in the presence of higher-curvature bulk corrections. We will also show and discuss the ambiguities that arise when doing so. These ambiguities are not present when the higher-curvature corrections are perturbative, and in this regime we will be able to introduce a novel, more compact expression to compute the entanglement entropy.

4.1 Holographic entanglement entropy functional in higher-curvature gravity

We will introduce in this section the holographic entanglement entropy functional when higher-curvature terms are present in the bulk action. To do so, it is useful to present a quick review of the construction that led [34] to a proof of the Ryu-Takayanagi formula, (4.2), because the method can be conveniently adapted to higher-curvature theories.

4.1.1 The Lewkowycz-Maldacena construction

Assume we have a certain spatial region of a field theory A, whose state is characterized by the density matrix ρ_A. The n-th Rényi entropy for $n = 2, 3, \ldots$ is defined as:

$$S_n(A) \equiv -\frac{1}{n-1} \log \mathrm{Tr}_A \left(\rho_A^n\right) \; . \tag{4.3}$$

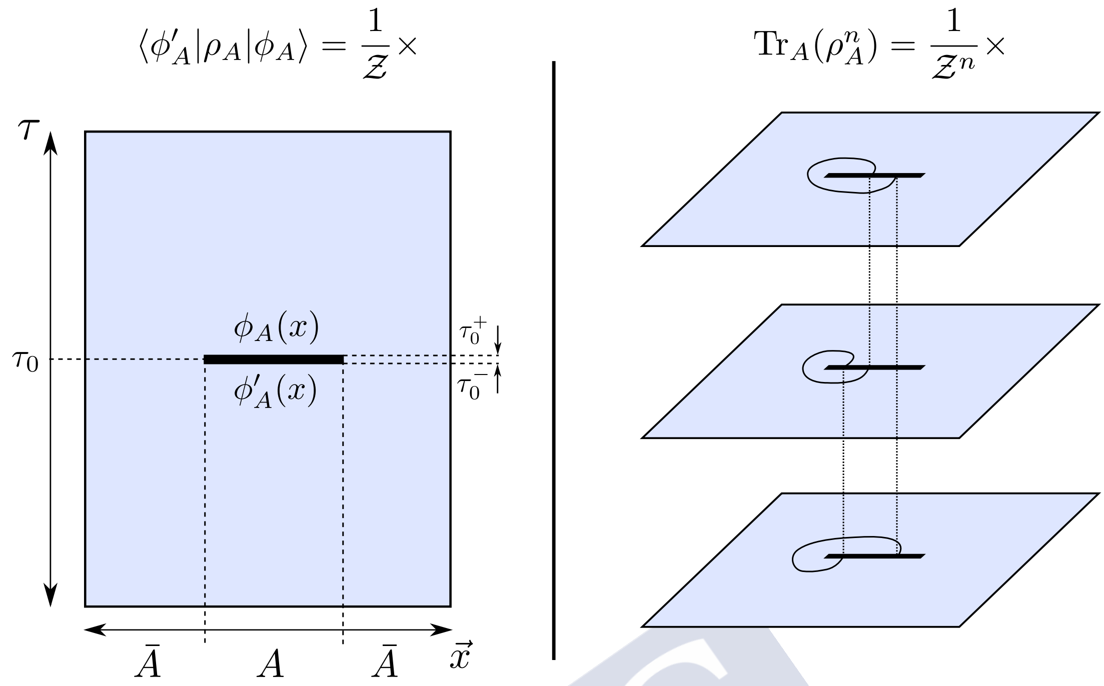

Figure 4.2: Replica trick representation of some important quantities when computing Rényi or entanglement entropies via path integrals. On the left, we see a pictorial representation of the reduced density matrix (4.5) when the system is in the vacuum state. We integrate over the whole Euclidean manifold (in blue), leaving open boundary conditions in region A in a fixed time slice $\tau = \tau_0$, corresponding to the bra and ket in the matrix element $\langle \phi'_A | \rho_A | \phi_A \rangle$. On the right, we replicate this construction to compute $\mathrm{Tr}_A(\rho_A^n)$ via an Euclidean manifold known as the n-fold cover, $\mathcal{M}_n$. The exact gluing procedure is provided by the form of the closed path drawn in the figure.

This object has the nice property that, if we analytically continue from integer n to any value and then take the limit as $n \to 1$, we obtain the the entanglement entropy of region A:

$$\lim_{n \to 1} S_n(A) = -\mathrm{Tr}_A \left(\rho_A \log \rho_A \right) = S_{\mathrm{EE}}(A) \ . \tag{4.4}$$

There is a nice field theory representation of the Rényi entropy (4.3) in terms of a path integral in a certain manifold, known as the n-fold cover, when we are in the vacuum state of the entire system. This is usually called the replica trick, [112], and it is illustrated in Figure 4.2. The idea is that the (unnormalized) density matrix is represented as an Euclidean path integral over the whole of spacetime $\mathcal{M}$, with open boundary conditions in region A. The correct normalization is obtained by dividing by $\mathcal{Z}$, the Euclidean partition function which corresponds to the integration on the whole Euclidean manifold:

$$\langle \phi'_A | \rho_A | \phi_A \rangle = \frac{1}{\mathcal{Z}} \int_{\mathcal{M}} [\mathcal{D}\phi(x)] \, e^{-\mathcal{I}_E[\phi]} \prod_{x \in A} \delta \left(\phi(x)|_+ - \phi_A(x) \right) \delta \left(\phi(x)|_- - \phi'_A(x) \right) \ , \tag{4.5}$$

where ϕ represents a generic field or set of fields, ϕ_A and ϕ'_A are field configuration states in region A, and the delta functions impose the correct open boundary conditions, which are probably better understood from the pictorial representation in Figure 4.2. The trace of the n-th power of ρ_A appearing in the definition of Rényi entropy is also neatly written in terms of a path integral, but this time in a n-times copied spacetimed glued along region A. We call this manifold the n-fold cover, $\mathcal{M}_n$, see once again Figure 4.2 to understand the exact gluing procedure. Define then the partition function on the replicated space as:

$$\mathcal{Z}_n \equiv \int_{\mathcal{M}_n} [\mathcal{D}\phi(x)]\, e^{-\mathcal{I}_E[\phi]} \;, \tag{4.6}$$

then, after analytic continuation in n, the entanglement entropy is given by:

$$S_{\rm EE}(A) = -\,\partial_n \left(\log \mathcal{Z}_n - n \log \mathcal{Z}\right)\big|_{n=1} \;. \tag{4.7}$$

Let us now rewrite the previous relation for a holographic theory, in which – in the semiclassical approximation – we can identify the partition function with the exponential of an on-shell bulk action, in the spirit of (4.1). If $\mathcal{B}$ is the bulk spacetime with boundary $\mathcal{M}$, while $\mathcal{B}_n$ is the dual to the n-fold cover $\mathcal{M}_n$:

$$S_{\rm EE}(A) = \partial_n \left(\mathcal{I}_E[\mathcal{B}_n] - n\mathcal{I}_E[\mathcal{B}]\right)\big|_{n=1} \;, \tag{4.8}$$

where $\mathcal{I}_E$ is the on-shell bulk Euclidean action. Notice that the boundary geometry $\mathcal{M}_n$ has a $\mathbb{Z}_n$ symmetry: if we define an angle θ going around the boundary of region A, the range of the coordinate is $2\pi n$ and there is a symmetry each time we shift θ by 2π. The dual bulk geometry $\mathcal{B}_n$ is a smooth solution of the bulk equations of motion, and θ naturally extends into the bulk. We assume this bulk geometry also preserves the $\mathbb{Z}_n$ symmetry, which corresponds to 2π shifts in θ. The fixed points of this $\mathbb{Z}_n$ action (which in the boundary $\mathcal{M}_n$ are exactly those in the boundary of region A) extend into the bulk, defining a codimension-2 surface $\Gamma_A^{(n)}$. We can quotient the bulk solution by the $\mathbb{Z}_n$ symmetry, so that we obtain an orbifold geometry $\hat{B}_n = B_n/\mathbb{Z}_n$ which is regular everywhere except at the codimension-2 surface of fixed points $\Gamma_A^{(n)}$. There, a conical singularity of opening angle $2\pi/n$ develops as a consequence of the identification. Once the quotient has been taken, we can analytically continue to non-integer values of n the definition of $\hat{\mathcal{B}}_n$. Furthermore, defining:

$$\hat{\mathcal{I}}_E[\hat{\mathcal{B}}_n] \equiv \mathcal{I}_E[\mathcal{B}_n]/n \;, \tag{4.9}$$

we can rewrite (4.8) as:

$$S_{\rm EE}(A) = \partial_n \hat{\mathcal{I}}_E[\hat{\mathcal{B}}_n]\Big|_{n=1} \;, \tag{4.10}$$

where we used $\lim_{n\to 1} \hat{\mathcal{I}}_E[\hat{\mathcal{B}}_n] = \mathcal{I}_E[\mathcal{B}]$.

Notice that $\hat{\mathcal{I}}_E[\hat{\mathcal{B}}_n]$ is not the on-shell action of the orbifold, $\mathcal{I}_E[\hat{\mathcal{B}}_n]$. This is a consequence of the conical singularity, which should not contribute to $\hat{\mathcal{I}}_E[\hat{\mathcal{B}}_n]$ because there is no such singular surface in $\mathcal{B}_n$, and we are just dividing by n in (4.9). On the contrary, the conical singularity contributes to the on-shell action of the singular manifold, $\mathcal{I}_E[\hat{\mathcal{B}}_n]$. This is a consequence of the δ-like behaviour of curvature tensors at the tip of the cone, a phenomenon which was studied in [113]. There, among other results, by regulating the conical singularity it is shown that:

$$\int_{\hat{\mathcal{B}}_n} \mathrm{d}^{d+1}x\sqrt{G}\, R = \frac{1}{n}\int_{\mathcal{B}_n} \mathrm{d}^{d+1}x\sqrt{G}\, R + 4\pi\left(1-\frac{1}{n}\right) \mathrm{Area}\left(\Gamma_A^{(n)}\right) \;. \tag{4.11}$$

In Einstein gravity (the case considered by Lewkowycz and Maldacena, plus possibly a cosmological constant which is not relevant for this discussion) this already relates the on-shell action for $\hat{\mathcal{B}}_n$ with the quantity (4.9) relevant for the entanglement entropy:

$$\mathcal{I}_E[\hat{\mathcal{B}}_n] = \hat{\mathcal{I}}_E[\hat{\mathcal{B}}_n] - \frac{1}{4G_N}\left(1-\frac{1}{n}\right)\text{Area}\left(\Gamma_A^{(n)}\right) . \tag{4.12}$$

Since $\hat{\mathcal{B}}_1 = \mathcal{B}$ is a solution of the bulk equations of motion, the variation of the on-shell action vanishes. In particular, $\partial_n \mathcal{I}_E[\hat{\mathcal{B}}_n]|_{n=1} = 0$, and from (4.10) this implies

$$S_{\text{EE}}^{\text{E}}(A) = \frac{\text{Area}(\Gamma_A^{(1)})}{4G_N} . \tag{4.13}$$

There only remains to show that $\Gamma_A^{(1)}$ is indeed the Γ_A in (4.2), which is to say that it has minimal area. This is done in [34] by considering the metric of the replicated bulk manifold $\mathcal{B}_n$ around the codimension-2 surface $\Gamma_A^{(n)}$ formed by the fixed points of $\mathbb{Z}_n$, and finding the constraints imposed by the equations of motion. To leading order in $(n-1)$, it can be shown that singularities in the equations of motion are avoided if:

$$K^a = 0 , \tag{4.14}$$

where K^a are the traces of the extrinsic curvatures along the transverse directions to $\Gamma_A^{(n)}$, and a is an index that runs in these two directions. The vanishing of the traces of the extrinsic curvatures guarantees that $\Gamma_A^{(n)}$ is a minimal area surface, and therefore in taking the limit $n \to 1$ we conclude $\Gamma_A^{(1)}$ is also so. Thus, (4.13) becomes the Ryu-Takayanagi formula, with $\Gamma_A^{(1)} = \Gamma_A$ the minimal area surface.

4.1.2 The splitting problem in higher-curvature gravity

The previous construction can be generalized to higher-curvature gravities, a task which was accomplished by Dong and Camps in [52, 53]. In fact, up until expression (4.10), we were being completly general. We must compute $\hat{\mathcal{I}}_E[\hat{\mathcal{B}}_n]$ for an arbitrary higher-curvature theory, and then take the n-derivative around $n = 1$. Recall that $\hat{\mathcal{I}}_E[\hat{\mathcal{B}}_n]$ is nothing but the on-shell orbifold action with the contribution from the conical singularity substracted – this is what relation (4.12) means in Einstein gravity. The presence of higher-curvature terms makes the details of the computation and its result way more cumbersome, even in terms of notation. We will gradually introduce the required definitions as we advance, but we remind the reader that we have collected our conventions in the Notation and conventions section for quick reference.

The key point when dealing with higher-curvature gravity is that equation (4.11) is not enough to relate the regularized action $\hat{\mathcal{I}}_E[\hat{\mathcal{B}}_n]$ with the on-shell version, $\mathcal{I}_E[\hat{\mathcal{B}}_n]$. Other terms apart from the Ricci scalar appear in the Lagrangian, and those contribute non-trivially at the conical singularity. The correct way to analyze this is by introducing a regularized metric for the orbifold $\hat{\mathcal{B}}_n$ but, to understand this procedure, we consider first the geometry before regularization. We set coordinates y^i for the $(d-1)$-dimensional singular surface $\Gamma_A^{(n)}$, ρ is a radial coordinate pointing away from this surface (located at $\rho = 0$), and τ is an angular coordinate running from 0 to 2π. For an opening angle of the conical singularity equal to $2\pi/n$, which is the case we are interested in, [52] showed that the metric can be written as:

$$\begin{aligned} \mathrm{d}s^2 =& \rho^{-2\epsilon}\left(\mathrm{d}z\mathrm{d}\bar{z}+\rho^{-2\epsilon}T\left(\bar{z}\mathrm{d}z-z\mathrm{d}\bar{z}\right)^2\right)+\left(h_{ij}+2K_{aij}x^a+\rho^{-2\epsilon}Q_{abij}x^ax^b\right)\mathrm{d}y^i\mathrm{d}y^j \\ &+2i\rho^{-2\epsilon}\left(U_i+V_{ai}\,x^a\right)\left(\bar{z}\mathrm{d}z-z\mathrm{d}\bar{z}\right)\mathrm{d}y^i+\dots\ , \end{aligned} \tag{4.15}$$

where the dots represent subleading terms when approaching the surface $\rho \to 0$, and we define $\epsilon = 1 - 1/n$. We have introduced here complex coordinates $z = \rho e^{i\tau}$, $\bar{z} = \rho e^{-i\tau}$ which help to simplify notation; and the x^a collectively denote these normal coordinates to the surface $(z, \bar{z})$. All coefficients appearing in the expansion (T, h_{ij}, K_{aij}, Q_{abij}, U_i, and V_{ai}) are themselves independient of x^a. Incidentally, K_{aij} are the extrinsic curvatures of the singular surface – this can be checked from the metric expansion. Notice that the metric close to $\rho = 0$ does have the structure of a conical singularity of opening angle $2\pi(1-\epsilon)$:

$$\mathrm{d}s^2\big|_{\rho\to 0} = \rho^{-2\epsilon}\mathrm{d}z\mathrm{d}\bar{z}+\dots = \frac{\mathrm{d}\rho^2}{\rho^{2\epsilon}}+\rho^{2(1-\epsilon)}\mathrm{d}\tau^2+\dots = \mathrm{d}\tilde{\rho}^2+(1-\epsilon)^2\tilde{\rho}^2\mathrm{d}\tau^2+\dots\ , \tag{4.16}$$

where we introduce the coordinate $\tilde{\rho} = \rho^{1-\epsilon}/(1-\epsilon)$. The locally polar angle would be $(1-\epsilon)\tau$, which only ranges from 0 to $2\pi(1-\epsilon)$, producing the conical singularity. An easy way to regulate (4.15) is by changing

$$\rho^{-2\epsilon} \to \frac{1}{(\rho^2+a^2)^{\epsilon}} \equiv e^{2A}\ , \tag{4.17}$$

where we defined $A = -\epsilon \log\left(\rho^2 + a^2\right)/2$, and we take $a > 0$ as the regulator.[2] In Dong's original proposal, [52], this substitution was done to every factor of $\rho^{-2\epsilon}$, which is indeed a valid and minimal – in the sense of introducing no extra terms with respect to (4.15) – regularization prescription. However, as several works pointed out and clarified later [114–117], this regularization may not be a solution to the equations of motion of the particular theory at hand. The way one has to regularize for a general, higher-curvature gravity is not known, as it is an extremely involved task to deal with the equations of motion of one of those theories. One can ask, however, what is the situation in Einstein gravity. In that case, one must take a regularization of the form [115, 116]:

$$\begin{aligned} \mathrm{d}s^2 =& e^{2A}\left(\mathrm{d}z\mathrm{d}\bar{z}+T\left(\bar{z}\mathrm{d}z-z\mathrm{d}\bar{z}\right)^2\right)+\left(h_{ij}+2K_{aij}x^a+Q_{abij}x^ax^b\right)\mathrm{d}y^i\mathrm{d}y^j \\ &+2e^{2A}iU_i\left(\bar{z}\mathrm{d}z-z\mathrm{d}\bar{z}\right)\mathrm{d}y^i+\dots\ , \end{aligned} \tag{4.18}$$

where inside T and Q_{abij} there are terms which contain factors e^{2A} (as in the original Dong's proposal), but also terms which do not.

All in all, the quantities appearing in the previous regularized metric progate through the curvature tensors to the action, and it is possible to compute the contribution from the conical singularity for each curvature combination. This is a technically convoluted task, so we refer the reader to the original works [52, 53] for more details. After that, we can substract the contribution of those terms from the on-shell action of the orbifold

[2] The final result is independent of the regulator, as argued in [52], because at the end we will only compute the coefficient of a potentially logarithmic divergence, which is regulator independent.

much like we did in (4.12), and then the same argument presented there allows to reach the final form of the entanglement entropy functional. It reads:

$$S_{\mathrm{EE}}^{\mathcal{L}_E(\mathrm{Riem})}(A) = S_{\mathrm{Wald}} + S_{\mathrm{Anomaly}} \,, \tag{4.19}$$

where $\mathcal{L}_E(\mathrm{Riem})$ indicates we allow higher-curvature corrections which are arbitrary contractions of the Riemann tensor, but not of its derivatives.[3] The first term in the previous equation has a form closely related to the Wald entropy of black holes,

$$S_{\mathrm{Wald}} = 2\pi \int_{\Gamma_A} \mathrm{d}^{d-1}y \, \sqrt{h} \, \frac{\partial \mathcal{L}_E}{\partial R_{z\bar{z}z\bar{z}}} \,, \tag{4.20}$$

while the anomaly part is:

$$S_{\mathrm{Anomaly}} = 2\pi \int_{\Gamma_A} \mathrm{d}^{d-1}y \, \sqrt{h} \sum_{\alpha} \left(\frac{\partial^2 \mathcal{L}_E}{\partial R_{zizj} \partial R_{\bar{z}k\bar{z}l}} \right)_{\alpha} \frac{8 K_{zij} K_{\bar{z}kl}}{1+q_\alpha} \,. \tag{4.21}$$

The α sum together with the q_α factor indicate that a certain weighting procedure has to be performed, consequence of the different contributions of the terms appearing in (4.18). The second derivative of the Lagrangian will be a sum of terms which are monomials with different contractions of components of the Riemann tensor. These contractions are to be expanded in terms of their z and $\bar{z}$ indices, obtaining an expression of the second derivative of the Lagrangian involving only $R_{z\bar{z}z\bar{z}}$, $R_{z\bar{z}zi}$, $R_{z\bar{z}ij}$, $R_{zi\bar{z}j}$, R_{zizj}, R_{zijk}, R_{ijkl}, plus components related to these by complex conjugation of the indices.[4] After this is done, the regularization of the conical defect will provide a “splitting”: a rule to divide each of the previous components of the Riemann tensor schematically as

$$R_{\mathcal{M}I} = \tilde{R}_{\mathcal{M}I} + \mathcal{K}_{\mathcal{M}I} \,. \tag{4.23}$$

In this expression, $\mathcal{M}$ labels the different components of the Riemann tensor enumerated before, while I is a generalized index containing all the $i, j, k, \ldots$ indices of the particular component under consideration (which might be none). This expansion has to be performed in all the components of the Riemann tensor, and once this is done, each of the resulting monomials is labelled by α. The splitting provides also a value q_α for each $\mathcal{K}_{\mathcal{M}I}$. In each term we have a definite value of q_α, given by the sum of the values of all the $\mathcal{K}_{\mathcal{M}I}$ in that monomial. Expression (4.21) instructs us then to divide each term by $q_\alpha + 1$. Once this is done, we can eliminate the $\tilde{R}_{\mathcal{M}I}$ (which are auxiliary objects in this construction, they can actually be related to the quantities appearing in the regularization (4.18), [52]) in favor of the Riemann tensor components by using (4.23) again.

[3] For an extension in which covariant derivatives of the curvature tensor are allowed, see [115].

[4] Notice that components of the Ricci tensor and the Ricci scalar have to be expanded in terms of these basic objects as well. For instance, we would write

$$R_{z\bar{z}} = G^{MN} R_{zM\bar{z}N} = -2 R_{z\bar{z}z\bar{z}} + h^{ij} R_{zi\bar{z}j} \,. \tag{4.22}$$

The particular splitting coming from the Einstein gravity regularization is:

$$R_{z\bar{z}z\bar{z}} = \tilde{R}_{z\bar{z}z\bar{z}} - \frac{1}{8}\left(K^{aij}K_{aij} - K^a K_a\right) , \tag{4.24a}$$

$$R_{z\bar{z}ij} = \tilde{R}_{z\bar{z}ij} - 2K_{z[i|}{}^{k}K_{\bar{z}|j]k} , \tag{4.24b}$$

$$R_{zi\bar{z}j} = \tilde{R}_{zi\bar{z}j} - K_{zi}{}^{k}K_{\bar{z}jk} + K_{(z}K_{\bar{z})ij} , \tag{4.24c}$$

$$R_{ijkl} = \tilde{R}_{ijkl} - 2K_{ai[k}K^{a}{}_{l]j} , \tag{4.24d}$$

with the remaining components having a trivial splitting, *i.e.*, $\tilde{R}_{\mathcal{M}I} = 0$ for them. The values of q_α are: $q_\alpha = 1$ for any of the previous terms quadratic in extrinsic curvatures, $q_\alpha = 1$ for R_{zizj} (and its complex conjugate), and $q_\alpha = 1/2$ for R_{zijk} and $R_{z\bar{z}zi}$ (and their complex conjugates). In the end, this complicated procedure is nothing but a way to generate contributions to the holographic entanglement entropy functional containing higher and higher powers of the extrinsic curvature. This will become much clearer in subsequent sections, where explicit calculations will be presented.

Let us emphasize that, for a generic higher-curvature theory, the particular form of the splittings which would substitute (4.24) is not known. It should be derived from the corresponding equations of motion, looking for the conditions they impose in the regularization of the conical defect; but this is definitely a technically complicated task. The lack of this general form is what has been called in the literature the "splitting problem", and it was overlooked at first in the original works [52, 53]. However, there are some cases in which this issue does not appear, and it is worth highlighting them now. First of all, quadratic theories do not have splitting ambiguities; their functional was obtained in [113]. This is a consequence of the fact that, after the two derivatives in the anomaly term are taken, no curvature tensors are left to expand in (4.21). A similar situation occurs for $f(R)$ theories – which only depend on the Ricci scalar: in this case, the second derivative in (4.21) vanishes, so that in fact all entropy comes from the Wald piece. Finally, the last important example in which the splitting does not matter are Lovelock theories [118]. The particular property here is that the functional depends only on the intrinsic geometry of the surface, the explicit argument can be found in [52]. The resulting functional actually matches the Jacobson-Myers entropy formula derived for black holes in Lovelock theories, [119].

All the previous discussion has been directed towards summarizing the way in which the holographic entanglement entropy functional (4.19) is obtained. There is still the question of where it is to be evaluated, *i.e.*, what is Γ_A in (4.20) and (4.21). In principle, just like in Einstein gravity, this surface is determined as the $n \to 1$ limit of the set of fixed points of the $\mathbb{Z}_n$ symmetry in the bulk. According to [120], the net result of this procedure has to be equivalent to minimizing the resulting functional (4.19), with both the Wald and the anomaly part. Therefore, the detailed analysis of the equations of motion can be avoided.

4.1.3 An example: cubic functionals in the Einstein gravity splitting

Let us clarify the procedure described in the previous section in a particular example. We will obtain the holographic entanglement entropy functional for a generic cubic gravitational theory, using the splitting dictated by Einstein gravity, (4.24). The most general cubic Lagrangian containing only contractions of the Riemann curvature is:

$$\begin{aligned}\mathcal{L}_E =&\mu_8\, R^3 + \mu_7\, R_{MN}R^{MN}R + \mu_6\, R_M{}^N R_N{}^R R_R{}^M + \mu_5\, R^{MR}R^{NS}R_{MNRS}\\ &+ \mu_4\, R_{MNRS}R^{MNRS}R + \mu_3\, R^{MNR}{}_S R_{MNRT}R^{ST}\\ &+ \mu_2\, R^{MN}{}_{RS}R^{RS}{}_{LT}R^{LT}{}_{MN} + \mu_1\, R_M{}^R{}_N{}^S R_R{}^L{}_S{}^T R_L{}^M{}_T{}^N\ .\end{aligned} \tag{4.25}$$

The holographic entanglement entropy functional (4.19) is linear in $\mathcal{L}_E$, so we can analyze each contribution separately. We will do the computation for a particular cubic term in full detail, while for the rest we will just present the final result. Consider then the following Lagrangian:

$$\mathcal{L}_E = R_{MNRS}R^{MNRS}R\ . \tag{4.26}$$

For the Wald term, we need the following derivative:

$$\frac{\partial \mathcal{L}_E}{\partial R_{z\bar{z}z\bar{z}}} = -2\lambda R_{MNRS}R^{MNRS} + 2RR^{z\bar{z}z\bar{z}}\ . \tag{4.27}$$

If we want to use this expression in a situation in which we do not have at our disposal the set of (complex) coordinates adapted to the surface, as will generically be the case, we need to covariantize the last term. The metric in the surface is given by (4.18) in the limit $\epsilon \to 0$ and taking $z, \bar{z} = 0$. Thus:

$$\mathrm{d}s^2\big|_{\Gamma_A} = \mathrm{d}z\mathrm{d}\bar{z} + h_{ij}\mathrm{d}y^i\mathrm{d}y^j\ , \tag{4.28}$$

which allows us to rewrite:

$$R^{z\bar{z}z\bar{z}} = -4R^{z\bar{z}}{}_{z\bar{z}} = -2R^{ab}{}_{ab}\ , \tag{4.29}$$

where the antisymmetry of the Riemann tensor in the two pairs of indices has been taken into account, and we recall that $a, b, \ldots$ denote here the normal directions to the surface. The last term, in a covariant form, is therefore the contraction of the Riemann tensor with the normal metric $\perp_{MN}= G_{MN} - h_{MN}$:

$$R^{z\bar{z}z\bar{z}} = -2R^{ab}{}_{ab} = -2R^{MNRS}\perp_{MR}\perp_{NS}\ . \tag{4.30}$$

Consider now the anomaly term. The first step is to obtain the second derivative of the Lagrangian at the surface:

$$\frac{\partial^2 \mathcal{L}_E}{\partial R_{zizj}\partial R_{\bar{z}k\bar{z}l}} = 2h^{i(k}h^{l)j}R\ . \tag{4.31}$$

Now we must expand the Riemann tensor components following (4.24):

$$R = -8R_{z\bar{z}z\bar{z}} + 8h^{ij}R_{zi\bar{z}j} + h^{ik}h^{jl}R_{ijkl} = -8\tilde{R}_{z\bar{z}z\bar{z}} + 8h^{ij}\tilde{R}_{zi\bar{z}j} + h^{ik}h^{jl}\tilde{R}_{ijkl}\ . \tag{4.32}$$

We see that the extrinsic curvatures cancel out in the Ricci scalar expansion. Therefore, there are no components with non-zero q_α in the anomaly sum (4.21), and we obtain:

$$\sum_\alpha \left(\frac{R}{1+q_\alpha}\right)_\alpha = R\ . \tag{4.33}$$

This completes the α expansion. Notice that, in (4.31), the metric tensors are not affected by the expansion, and we can contract them with the extrinsic curvatures appearing in the anomaly term as follows:

$$h^{i(k}h^{l)j}K_{zij}K_{\bar{z}kl} = K_{zij}K_{\bar{z}}{}^{ij} = \frac{1}{4}K_{aij}K^{aij} \ . \tag{4.34}$$

We conclude this little example by collecting all contributions, which would produce the following entanglement entropy functional for the Lagrangian (4.26):

$$S_{\mathrm{EE}}(A) = 4\pi \int_{\Gamma_A} \mathrm{d}^{d-1}y\,\sqrt{h}\,\left[-R_{MNRS}R^{MNRS} - 2RR^{ab}{}_{ab} + 2K_{aij}K^{aij}R\right] \ . \tag{4.35}$$

We can follow an identical procedure to obtain the functional of the general cubic theory (4.25). The result can be written as:

$$S_{\mathrm{EE}}(A) = 2\pi \int_{\Gamma_A} \mathrm{d}^{d-1}y\,\sqrt{h}\,\left(S_{R^2} + S_{K^2R} + S_{K^4}\right) \ , \tag{4.36}$$

where

$$\begin{aligned}
S_{R^2} = &- 6\mu_8 R^2 - 2\mu_7\left(R_{MN}R^{MN} + R_a{}^aR\right) - 3\mu_6 R_{aM}R^{aM} \\
&- \mu_5\left(2R_{MN}R_a{}^{MaN} - R_{ab}R^{ab} + R_a{}^aR_b{}^b\right) - 2\mu_4\left(R_{MNRS}R^{MNRS} + 2RR_{ab}{}^{ab}\right) \\
&- \mu_3\left(R_{aMNR}R^{aMNR} + 4R_{aM}R_b{}^{abM}\right) - 6\mu_2 R_{abMN}R^{abMN} \\
&+ 3\mu_1\left(R_{aMbN}R^{aNbM} - R_{aM}{}^a{}_NR_b{}^{MbN}\right) ,
\end{aligned} \tag{4.37a}$$

$$\begin{aligned}
S_{K^2R} = &+ \mu_7 K_aK^aR + \frac{3}{2}\mu_6K_aK^aR_b{}^b + 2\mu_5K_aK^a{}_{ij}R^{ij} - \frac{1}{2}\mu_5K_aK^aR_{bc}{}^{bc} + 4\mu_4K_{aij}K^{aij}R \\
&+ 2\mu_3K_{aik}K^a{}_j{}^kR^{ij} + \mu_3K_{aij}K^{aij}R_b{}^b + 2\mu_3K_aK^a{}_{ij}R_b{}^{ibj} + 12\mu_2K_{aik}K^a{}_j{}^kR_b{}^{ibj} \\
&+ 3\mu_1K_{aij}K^a{}_{kl}R^{ikjl} - \frac{3}{2}\mu_1K_{aij}K^{aij}R_{bc}{}^{bc} + 6\left(2\mu_2 + \mu_1\right)K_{aik}K_{bj}{}^kR^{abij} \ ,
\end{aligned} \tag{4.37b}$$

$$\begin{aligned}
S_{K^4} = &+ \frac{1}{4}\left(\mu_5 - 3\mu_1\right)K_aK^aK_{bij}K^{bij} - \frac{1}{4}\mu_5K_aK^aK_bK^b + \left(\mu_3 - 6\mu_2\right)K_aK^a{}_{ij}K_b{}^i{}_kK^{bjk} \\
&- \mu_3K_aK^a{}_{ij}K_bK^{bij} + \frac{3}{4}\mu_1K_{aij}K^{aij}K_{bkl}K^{bkl} + \frac{3}{2}\mu_1K_{aij}K_{bkl}K^{bij}K^{akl} \\
&- \frac{3}{2}\left(4\mu_2 + 3\mu_1\right)K_{ai}{}^jK_{bj}{}^kK^a{}_k{}^lK^b{}_l{}^i + 3\left(4\mu_2 + \mu_1\right)K_{ai}{}^jK^a{}_j{}^kK_{bk}{}^lK^b{}_l{}^i \ .
\end{aligned} \tag{4.37c}$$

We can explicitly see in these expressions how the procedure described in the previous section has generated terms with different powers of the extrinsic curvature (K^2 and K^4) in the holographic entanglement entropy functional.

We emphasize once again that this is the result obtained by means of the Einstein gravity splitting, (4.24), applied to the generic cubic Lagrangian. In principle, for a given cubic theory, we would have to analyze the equations of motion for the regularized conical defect geometry, and find how the constraints they impose translate into a particular recipe for the α expansion in the anomaly term of the holographic entanglement entropy functional. It could be that the previous formula is valid, but it might also happen that it has to be corrected due to a different splitting. Furthermore, in a particular calculation of the entanglement entropy of a certain boundary region A, even if (4.36) is valid, we would have to minimize it, and the higher order terms would produce Euler-Lagrange equations for the minimization problem which are not second order in derivatives. In this situation, it is not completely clear how to deal with the boundary value problem of finding a bulk surface Γ_A which minimizes $S_{\rm EE}(A)$ subject to the constraint that its boundary coincides with that of region A. For this reason, from now on we will follow a different approach, which was the one presented in [5]. If we work perturbatively in the higher-curvature couplings, things become much more tractable. First of all, the splitting is unambiguously set to be the Einstein gravity one. This is because the higher-curvature terms would produce modifications of (4.24) proportional to the couplings. This would manifest itself in the α expansion of the anomaly term, but since the second derivative of the Lagrangian is already first order in the couplings in (4.21), the modification of the α expansion produces subleading terms which can be discarded to leading order. Furthermore, the minimization problem is also easy to solve perturbatively. Schematically, the functional will look like:

$$S_{\rm EE}(A) = \frac{\text{Area}(\Gamma_A)}{4G_N} + \lambda S_{\rm corr}(\Gamma_A) \, , \tag{4.38}$$

where λ is a generic coupling, and we collect all the corrections in $S_{\rm corr}(\Gamma_A)$, which can be calculated with the Einstein gravity splitting, as we have just argued. The Euler-Lagrange equations derived from asking Γ_A to be a minimal surface of the previous functional are solved by a surface which is the RT one, minimizing the area term, with a first order perturbative correction. Let Γ_A be this solution, and $\Gamma_{\rm RT}$ be the RT minimal-area surface. Since the area is stationary for the RT surface, $\text{Area}(\Gamma_A) = \text{Area}(\Gamma_{\rm RT}) + \mathcal{O}(\lambda^2)$. Furthermore, since the corrections are already multiplied by λ, we also have $\lambda S_{\rm corr}(\Gamma_A) = \lambda S_{\rm corr}(\Gamma_{\rm RT}) + \mathcal{O}(\lambda^2)$. Therefore, to leading order in λ, we can directly evaluate the entanglement entropy functional in the RT surface, which is defined to have minimal area, and thus satisfies $K^a = 0$. This simplifies a lot many of the expressions obtained for the higher-curvature holographic entanglement entropy functionals. In the following sections, working in this perturbative setup, we will obtain a rewriting of the general functional which converts the anomaly term to a neater and computationally simpler form.

4.2 Rewriting the HEE functional

In this section, we will present a convenient rewriting of the holographic entanglement entropy functional first introduced in [5]. Its main advantage when compared to (4.19) is that we will get rid of the α sum in the anomaly term. This makes the resulting expression easier to interpret – while (4.21) falsely seems to be quadratic in extrinsic curvatures due to the α sum, our rewriting will clarify the process by which K-terms are generated –, but it also makes it more amenable to explicit computations. This, in turn, will allow us to develop a general understanding of the structure of the functional for different kind of theories in subsequent sections. In what follows, we will have in the back of our mind the Einstein gravity splitting (4.24) but in the perturbative approximation, in which we will evaluate at the RT surface and therefore we can assume $K^a = 0$. That means our splitting is:

$$
\begin{aligned}
R_{z\bar{z}z\bar{z}} &= \tilde{R}_{z\bar{z}z\bar{z}} - \frac{1}{8}K^{aij}K_{aij} \,, && \text{(4.39a)}\\
R_{z\bar{z}ij} &= \tilde{R}_{z\bar{z}ij} - 2K_{z[i|}{}^{k}K_{\bar{z}|j]k} \,, && \text{(4.39b)}\\
R_{zi\bar{z}j} &= \tilde{R}_{zi\bar{z}j} - K_{zi}{}^{k}K_{\bar{z}jk} \,, && \text{(4.39c)}\\
R_{ijkl} &= \tilde{R}_{ijkl} - 2K_{ai[k}K^{a}{}_{l]j} \,. && \text{(4.39d)}
\end{aligned}
$$

However, the derivation leading to the rewriting of the functional will be pretty general, so in principle we expect the final form obtained to be valid even for different splittings. It would be applicable, in particular, to the non-perturbative Einstein splitting (4.24). The only requirement is that the Riemann tensor components are splitted as in (4.23),

$$
R_{\mathcal{M}I} = \tilde{R}_{\mathcal{M}I} + \mathcal{K}_{\mathcal{M}I} \,, \tag{4.40}
$$

with $\mathcal{K}_{\mathcal{M}I}$ having a definite value for q_α.

Let us now introduce some notation which will simplify our arguments. First of all, notice that the conventional definition of the derivative with respect to the Riemann tensor is:

$$
\frac{\partial R_{MNPQ}}{\partial R_{RSTU}} \equiv \frac{1}{2}\left[\delta^{R}_{[M}\delta^{S}_{N]}\delta^{T}_{[P}\delta^{U}_{Q]} + \delta^{R}_{[P}\delta^{S}_{Q]}\delta^{T}_{[M}\delta^{U}_{N]}\right] \,. \tag{4.41}
$$

This definition respects the symmetries of the Riemann tensor and, at the same time, it has the following nice (and expected) property,

$$
R_{RSTU}\frac{\partial R_{MNPQ}}{\partial R_{RSTU}} = R_{MNPQ} \,, \tag{4.42}
$$

which will be key when performing Taylor-like expansions of functions of the Riemann tensor. But this definition conflicts with the process of singling out specific components, due to the symmetries. For instance, using the previous definition in the coordinates adapted to the bulk entangling surface, one finds:

$$
\frac{\partial R_{z\bar{z}ij}}{\partial R_{z\bar{z}kl}} = \frac{1}{2}\left[\delta^{z}_{[z}\delta^{\bar{z}}_{\bar{z}]}\delta^{k}_{[i}\delta^{l}_{j]} + \delta^{z}_{[i}\delta^{\bar{z}}_{j]}\delta^{k}_{[z}\delta^{l}_{\bar{z}]}\right] = \frac{1}{4}\delta^{k}_{[i}\delta^{l}_{j]} \,, \tag{4.43}
$$

which leads to

$$
R_{z\bar{z}kl}\frac{\partial R_{z\bar{z}ij}}{\partial R_{z\bar{z}kl}} = \frac{1}{4}R_{z\bar{z}ij} \,. \tag{4.44}
$$

The factor $1/4$ arises from the different positions in which we can put the z, $\bar{z}$ indices using the symmetries of the Riemann tensor, $R_{z\bar{z}kl}$, $R_{\bar{z}zkl}$, $R_{klz\bar{z}}$, and $R_{kl\bar{z}z}$. Something analogous happens for the remaining components of the Riemann tensor. Hence, whenever performing Taylor-like expansions, we will need to take these extra factors into account. To avoid clutter, we define a new derivative operator, $\hat{\partial}$, which already includes them:

$$
\begin{aligned}
&\frac{\hat{\partial}}{\hat{\partial}R_{z\bar{z}z\bar{z}}} \equiv 4\frac{\partial}{\partial R_{z\bar{z}z\bar{z}}} \,, \qquad
\frac{\hat{\partial}}{\hat{\partial}R_{z\bar{z}zi}} \equiv 8\frac{\partial}{\partial R_{z\bar{z}zi}} \,, \qquad
\frac{\hat{\partial}}{\hat{\partial}R_{z\bar{z}ij}} \equiv 4\frac{\partial}{\partial R_{z\bar{z}ij}} \,,\\
&\frac{\hat{\partial}}{\hat{\partial}R_{zizj}} \equiv 4\frac{\partial}{\partial R_{zizj}} \,, \qquad
\frac{\hat{\partial}}{\hat{\partial}R_{zi\bar{z}j}} \equiv 8\frac{\partial}{\partial R_{zi\bar{z}j}} \,, \qquad
\frac{\hat{\partial}}{\hat{\partial}R_{zijk}} \equiv 4\frac{\partial}{\partial R_{zijk}} \,,\\
&\frac{\hat{\partial}}{\hat{\partial}R_{ijkl}} \equiv \frac{\partial}{\partial R_{ijkl}} \,.
\end{aligned} \tag{4.45}
$$

The remaining components can be obtained from these by complex conjugation.

Another piece of notation we need is that aimed at collectively manipulating the different components of the Riemann tensor. Recall that upper case latin indices $I, J, \ldots$ represent all $i, j, k, \ldots$ indices that might appear in a given tensor. For instance, in $R_{z\bar{z}ij}$, an index I would refer to ij. Similarly, we introduce $\mathcal{M}, \mathcal{N}, \ldots$ indices to represent the different Riemann tensor components involving z and $\bar{z}$ indices. In practice, we just want this notation to perform Taylor expansions, for which the relevant thing to keep in mind is the following compact definition:

$$\begin{aligned} R_{\mathcal{M}I}\hat{\partial}^{\mathcal{M}I} \equiv &+ R_{z\bar{z}z\bar{z}}\frac{\hat{\partial}}{\hat{\partial}R_{z\bar{z}z\bar{z}}} + R_{z\bar{z}ij}\frac{\hat{\partial}}{\hat{\partial}R_{z\bar{z}ij}} + R_{zi\bar{z}j}\frac{\hat{\partial}}{\hat{\partial}R_{zi\bar{z}j}} + R_{ijkl}\frac{\hat{\partial}}{\hat{\partial}R_{ijkl}} \\ &+ \left[R_{z\bar{z}zi}\frac{\hat{\partial}}{\hat{\partial}R_{z\bar{z}zi}} + R_{zizj}\frac{\hat{\partial}}{\hat{\partial}R_{zizj}} + R_{zijk}\frac{\hat{\partial}}{\hat{\partial}R_{zijk}} + \text{c.c.}\right] , \end{aligned} \tag{4.46}$$

where c.c. stands for the complex conjugate components of the terms in the parentheses (which are the only ones that have a different number of z and $\bar{z}$ indices). This can be thought of as a sum over $\mathcal{M}$ (the z and $\bar{z}$ indices) and then, for each $\mathcal{M}$, an extra sum over tangent indices I. This is useful because different components of the Riemann tensor have different splitting structures. In general, any component splits as in (4.40), where $\tilde{R}_{\mathcal{M}I}$ has $q_\alpha = 0$ and $\mathcal{K}_{\mathcal{M}I}$ has $q_\alpha \neq 0$. The q_α for the $\mathcal{K}_{\mathcal{M}I}$ piece can take two values. Components $R_{z\bar{z}z\bar{z}}, R_{z\bar{z}ij}, R_{zi\bar{z}j}, R_{zizj}, R_{\bar{z}i\bar{z}j}$, and R_{ijkl} have $q_\alpha = 1$ for that part, and we will generically refer to them with labels $\mathcal{A}, \mathcal{A}', \ldots$ On the other hand, components $R_{zijk}, R_{\bar{z}ijk}, R_{z\bar{z}zi}$, and $R_{\bar{z}z\bar{z}i}$ have $q_\alpha = 1/2$ for the $\mathcal{K}_{\mathcal{M}I}$ part, and we will refer to them with labels $\mathcal{B}, \mathcal{B}', \ldots$. In terms of these, the operator (4.46) splits into two contributions:

$$\begin{aligned} R_{\mathcal{A}I}\hat{\partial}^{\mathcal{A}I} \equiv &+ R_{z\bar{z}z\bar{z}}\frac{\hat{\partial}}{\hat{\partial}R_{z\bar{z}z\bar{z}}} + R_{z\bar{z}ij}\frac{\hat{\partial}}{\hat{\partial}R_{z\bar{z}ij}} + R_{zi\bar{z}j}\frac{\hat{\partial}}{\hat{\partial}R_{zi\bar{z}j}} + R_{ijkl}\frac{\hat{\partial}}{\hat{\partial}R_{ijkl}} \\ &+ \left[R_{zizj}\frac{\hat{\partial}}{\hat{\partial}R_{zizj}} + \text{c.c.}\right] , \end{aligned} \tag{4.47a}$$

$$R_{\mathcal{B}I}\hat{\partial}^{\mathcal{B}I} \equiv \left[R_{zijk}\frac{\hat{\partial}}{\hat{\partial}R_{zijk}} + R_{z\bar{z}zi}\frac{\hat{\partial}}{\hat{\partial}R_{z\bar{z}zi}} + \text{c.c.}\right] . \tag{4.47b}$$

To complete the rewriting of the anomaly part of the holographic entanglement entropy functional, we define also the following shorthand notation for the object on which the α expansion is to be performed:

$$\left(\frac{8\partial^2\mathcal{L}_E}{\partial\text{Riem}^2}K^2\right) \equiv 8\frac{\partial^2\mathcal{L}_E}{\partial R_{zizj}\partial R_{\bar{z}k\bar{z}l}}K_{zij}K_{\bar{z}kl} \, . \tag{4.48}$$

Equipped with the previous notation, we can proceed to rewrite the anomaly term of the holographic entanglement entropy functional. We relegate to appendix B the technical details of this process, and we quote here only the final result:

$$\begin{aligned}&\sum_\alpha \frac{1}{1+q_\alpha}\left(\frac{8\partial^2\mathcal{L}_E}{\partial\mathrm{Riem}^2}K^2\right)_\alpha \\ &=\sum_{S=0}^{\infty}\frac{1}{S!}\int_0^1 \mathrm{d}u\, 2u : \left[-(1-u^2)\mathcal{K}_{\mathcal{A}I}\hat{\partial}^{\mathcal{A}I}-(1-u)\mathcal{K}_{\mathcal{B}J}\hat{\partial}^{\mathcal{B}J}\right]^S : \left(\frac{8\partial^2\mathcal{L}_E}{\partial\mathrm{Riem}^2}K^2\right) .\end{aligned} \tag{4.49}$$

In this expression, $\mathcal{K}_{\mathcal{A}I}\hat{\partial}^{\mathcal{A}I}$ and $\mathcal{K}_{\mathcal{B}J}\hat{\partial}^{\mathcal{B}J}$ are given by the splittings (4.40) of the type-A ($q_\alpha = 1$) and type-B ($q_\alpha = 1/2$) components. Explicitly, in the perturbative Einstein gravity splitting, (4.39):

$$\begin{aligned}\mathcal{K}_{\mathcal{A}I}\hat{\partial}^{\mathcal{A}I} \equiv & -\frac{1}{2}K^{aij}K_{aij}\frac{\partial}{\partial R_{z\bar{z}z\bar{z}}} - 8K_{zi}{}^{k}K_{\bar{z}jk}\frac{\partial}{\partial R_{z\bar{z}ij}} - 8K_{zi}{}^{k}K_{\bar{z}jk}\frac{\partial}{\partial R_{zi\bar{z}j}} \\ & - 2K_{aik}K^{a}{}_{lj}\frac{\partial}{\partial R_{ijkl}} + \left(4R_{zizj}\frac{\partial}{\partial R_{zizj}} + \text{c.c.}\right) ,\end{aligned} \tag{4.50a}$$

$$\mathcal{K}_{\mathcal{B}J}\hat{\partial}^{\mathcal{B}J} \equiv \left(4R_{zijk}\frac{\partial}{\partial R_{zijk}} + 8R_{z\bar{z}zi}\frac{\partial}{\partial R_{z\bar{z}zi}} + \text{c.c.}\right) . \tag{4.50b}$$

Furthermore, the normal ordering $::$ in (4.49) means that derivatives are to be moved to the right before taking them, for instance:

$$:\left(R_{zizj}\frac{\partial}{\partial R_{zizj}}\right)^2 := R_{zizj}R_{zkzl}\frac{\partial^2}{\partial R_{zizj}\partial R_{zkzl}} . \tag{4.51}$$

Observe that the sum in (4.49) can be formally performed, allowing us to write the result in an exponential form:

$$\sum_\alpha \frac{1}{1+q_\alpha}\left(\frac{8\partial^2\mathcal{L}_E}{\partial\mathrm{Riem}^2}K^2\right)_\alpha = \int_0^1 \mathrm{d}u\, 2u\, : e^{-F(u)} : \left(\frac{8\partial^2\mathcal{L}_E}{\partial\mathrm{Riem}^2}K^2\right) , \tag{4.52}$$

where

$$F(u) \equiv \left[(1-u^2)\mathcal{K}_{\mathcal{A}I}\hat{\partial}^{\mathcal{A}I} + (1-u)\mathcal{K}_{\mathcal{B}J}\hat{\partial}^{\mathcal{B}J}\right] . \tag{4.53}$$

4.2.1 A simple example mixing type-A and type-B terms

To exemplify how the previous formula works in a particular example, let us consider a particular computation of the anomaly piece. Assume we have a fifth-order Lagrangian which produces, after two derivatives and contractions as in (4.48), the following term as part of the whole expression:[5]

$$\left(\frac{8\partial^2\mathcal{L}_E}{\partial\mathrm{Riem}^2}K^2\right) \supset C(K^2)R_{zijk}R_{\bar{z}z\bar{z}}{}^{k}R_{z}{}^{i}{}_{\bar{z}}{}^{j} , \tag{4.54}$$

[5]We have not checked if this terms appears in a particular fifth-order functional, but for illustrative purposes this does not matter. It certainly could, because the only important restriction is that (4.48) produces terms with equal number of z and $\bar{z}$, and this is respected by our example.

where $C(K^2) \equiv cK_z{}^{lm}K_{\bar{z}lm}$ with c a constant, and the $\supset$ symbol means that this is only one of many terms that would appear when expanding the second derivative in terms of the different z and $\bar{z}$ components of the curvature tensor for an actual Lagrangian.

Let us first obtain the contribution to the anomaly piece of the functional by means of the α sum, which in this case turns out to be particularly simple. Applying the splitting rules (4.39), this term becomes:

$$\sum_\alpha \left(\frac{8\partial^2\mathcal{L}_E}{\partial\text{Riem}^2}K^2\right)_\alpha \supset C(K^2)\left(R_{zijk}R_{\bar{z}z\bar{z}}{}^k\tilde{R}_z{}^i{}_{\bar{z}}{}^j - R_{zijk}R_{\bar{z}z\bar{z}}{}^kK_z{}^{il}K_{\bar{z}}{}^j{}_l\right). \tag{4.55}$$

The first term has $q_\alpha = 1$, while the second has $q_\alpha = 2$. Then, dividing by $1+q_\alpha$, we get:

$$\begin{aligned}\sum_\alpha \frac{1}{1+q_\alpha}\left(\frac{8\partial^2\mathcal{L}_E}{\partial\text{Riem}^2}K^2\right)_\alpha &\supset C(K^2)\left(\frac{1}{2}R_{zijk}R_{\bar{z}z\bar{z}}{}^k\tilde{R}_z{}^i{}_{\bar{z}}{}^j - \frac{1}{3}R_{zijk}R_{\bar{z}z\bar{z}}{}^kK_z{}^{il}K_{\bar{z}}{}^j{}_l\right)\\ &= C(K^2)\left(\frac{1}{2}R_{zijk}R_{\bar{z}z\bar{z}}{}^kR_z{}^i{}_{\bar{z}}{}^j + \frac{1}{6}R_{zijk}R_{\bar{z}z\bar{z}}{}^kK_z{}^{il}K_{\bar{z}}{}^j{}_l\right),\end{aligned} \tag{4.56}$$

where we have rewritten $\tilde{R}_z{}^i{}_{\bar{z}}{}^j$ in terms of the Riemann tensor component again in the last line.

Let us now obtain the same result by means of the rewriting of the anomaly piece in terms of the derivative expression. We need to take into account terms up to $S=3$ in the series, but fortunately not every type A or B component appears in the piece of the Lagrangian we are considering. This means we can define new operators including only the relevant parts:

$$\partial_A \equiv -8K_{zi}{}^lK_{\bar{z}jl}\frac{\partial}{\partial R_{zi\bar{z}j}}, \qquad \partial_B \equiv 4R_{zijk}\frac{\partial}{\partial R_{zijk}} + 8R_{\bar{z}z\bar{z}k}\frac{\partial}{\partial R_{\bar{z}z\bar{z}k}}. \tag{4.57}$$

The $S=0$ term is just the original (4.54). For the $S=1$ term we apply the operator:

$$-\frac{1}{2!}\partial_A - \frac{2}{3!}\partial_B, \tag{4.58}$$

which produces:

$$s_1 \equiv C(K^2)\left(-\frac{2}{3}R_{zijk}R_{\bar{z}z\bar{z}}{}^kR_z{}^i{}_{\bar{z}}{}^j + \frac{1}{2}R_{zijk}R_{\bar{z}z\bar{z}}{}^kK_z{}^{il}K_{\bar{z}}{}^j{}_l\right). \tag{4.59}$$

For the $S=2$ term operator we already find mixing between ∂_A and ∂_B. Solving the integral expression given in (4.49),

$$\frac{1}{2!}\int_0^1 \mathrm{d}u\, 2u : \left(-(1-u^2)\partial_A - (1-u)\partial_B\right)^2 : = : \left(\frac{1}{6}\partial_A^2 + \frac{7}{30}\partial_A\partial_B + \frac{1}{12}\partial_B^2\right) : . \tag{4.60}$$

We stress once again that normal ordering means that derivatives do not act on curvature components appearing in the operators (4.57), only on those components in the second derivative object (4.54). This makes ∂_A and ∂_B commuting objects (inside a normal ordered expression). Furthermore, having only a single type A component, the ∂_A^2 term in the previous expression will not contribute. The last two terms produce:

$$s_2 \equiv C(K^2)\left(\frac{1}{6}R_{zijk}R_{\bar{z}z\bar{z}}{}^kR_z{}^i{}_{\bar{z}}{}^j - \frac{7}{15}R_{zijk}R_{\bar{z}z\bar{z}}{}^kK_z{}^{il}K_{\bar{z}}{}^j{}_l\right). \tag{4.61}$$

Finally, let us consider the $S=3$ term. The operator is

$$\frac{1}{3!}\int_0^1 \mathrm{d}u\, 2u : \left(-(1-u^2)\partial_A - (1-u)\partial_B\right)^3 : =$$
$$= - : \left(\frac{\partial_A^3}{24} + \frac{19\partial_A^2\partial_B}{210} + \frac{\partial_A\partial_B^2}{15} + \frac{\partial_B^3}{60}\right) : . \qquad (4.62)$$

In this case, since (4.54) has one type A and two type B terms, the third piece of this operator is the only one giving a non-vanishing contribution. Its value is:

$$s_3 \equiv \frac{2}{15} C(K^2) R_{zijk} R_{\bar{z}z\bar{z}}{}^{k} K_z{}^{il} K_{\bar{z}}{}^{j}{}_{l} \,. \qquad (4.63)$$

Collecting all contributions s_0, s_1, s_2, and s_3:

$$\sum_\alpha \frac{1}{1+q_\alpha}\left(\frac{8\partial^2\mathcal{L}_E}{\partial \mathrm{Riem}^2} K^2\right)_\alpha$$
$$\supset C(K^2)\left(\frac{1}{2} R_{zijk} R_{\bar{z}z\bar{z}}{}^{k} R_z{}^{i}{}_{\bar{z}}{}^{j} + \frac{1}{6} R_{zijk} R_{\bar{z}z\bar{z}}{}^{k} K_z{}^{il} K_{\bar{z}}{}^{j}{}_{l}\right) , \qquad (4.64)$$

which coincides with (4.56), as it should. Notice that, in this particular example, the derivative expansion does not seem to improve the α sum algorithm to reach the final contribution to the anomaly piece: the result (4.56) was more easily obtained by the latter procedure. This is because the particular form of the term chosen, (4.54), was easy to expand, and already defined in terms of the z, $\bar{z}$ components of the Riemann tensors. For a general, realistic Lagrangian this is not the case, and in particular implementing computationally the derivative sum (4.49) is way easier than doing it for the α expansion.

4.2.2 Covariant form of the functional

So far we have presented all our expressions in the particular set of adapted coordinates $(z, \bar{z}, y^i)$. Here we will rewrite our general formulas in covariant form, which sometimes can be more useful for explicit applications. In order to do that, we first introduce two orthonormal vectors to the bulk entanglement surface Γ_A, $n_a{}^M$, such that $G_{MN} n_a{}^M n_b{}^N = \delta_{ab}$ (we work in Euclidean signature). Defining $n^a{}_M \equiv \delta^{ab} n_b{}^N G_{NM}$, the metric in the bulk entangling surface can be decomposed as:

$$G_{MN} = h_{MN} + \delta_{ab} n^a{}_M n^b{}_N \,, \qquad (4.65)$$

so that in the adapted coordinates $n^a{}_i = 0$, and h_{MN} is non-vanishing only for tangent components ($h_{zz} = h_{z\bar{z}} = h_{\bar{z}\bar{z}} = 0$). We also define the binormal to the surface and the normal projector, respectively, as:

$$\epsilon_{MN} \equiv \epsilon_{ab} n^a{}_M n^b{}_N \,, \qquad \perp_{MN} \equiv \delta_{ab} n^a{}_M n^b{}_N \,, \qquad (4.66)$$

where ϵ_{ab} is the two-dimensional Levi-Civita symbol. It is easy to check that, in the adapted coordinates, $\epsilon_{z\bar{z}} = -\epsilon_{\bar{z}z} = i/2$, $\perp_{zz}=\perp_{\bar{z}\bar{z}}= 0$,[6] and $\perp_{z\bar{z}}= 1/2$. This implies

$$\delta^z_M \delta^{\bar{z}}_N = \perp_{MN} - i\epsilon_{MN} \,, \qquad \delta^M_z \delta^N_{\bar{z}} = \frac{1}{4}\left(\perp^{MN} + i\epsilon^{MN}\right) \,, \tag{4.67a}$$

$$\delta^z_M \delta^N_z = \frac{1}{2}\left(\perp_M{}^N - i\epsilon_M{}^N\right) \,, \qquad \delta^{\bar{z}}_M \delta^N_{\bar{z}} = \frac{1}{2}\left(\perp_M{}^N + i\epsilon_M{}^N\right) \,. \tag{4.67b}$$

These are all different forms of the same identity, related by raising or lowering the z and $\bar{z}$ indices, but the different forms are useful in different contexts. In particular, they can be used to write in a covariant form the different terms appearing in the entanglement entropy functional.

Let us start with the Wald term,

$$\frac{\partial \mathcal{L}_E}{\partial R_{z\bar{z}z\bar{z}}} = \delta^z_{[M}\delta^{\bar{z}}_{N]}\delta^z_{[R}\delta^{\bar{z}}_{S]} \frac{\partial \mathcal{L}_E}{\partial R_{MNRS}} = -\epsilon_{MN}\epsilon_{RS} \frac{\partial \mathcal{L}_E}{\partial R_{MNRS}} \,. \tag{4.68}$$

The last form, which is the familiar one for this piece [51, 84], is fully covariant, as desired. Similar manipulations can be applied to the anomaly term. We consider the extrinsic curvatures as part of a spacetime tensor:

$$K^a{}_{MN} \equiv h^R_M h^S_N \nabla_R n^a{}_S \,, \qquad K^L{}_{MN} \equiv K^a{}_{MN} n_a{}^L \,, \tag{4.69}$$

which satisfies, in adapted coordinates, $K^L{}_{MN} V^N = K^L{}_{Mi} V^i$ for any vector V^N. Then, for the second derivative of the Lagrangian contracted with two extrinsic curvatures we get:

$$\begin{aligned}\left(\frac{8\partial^2 \mathcal{L}_E}{\partial \mathrm{Riem}^2} K^2\right) =& 2\left[\perp^{L_1L_2}\left(\perp_{M_1M_2}\perp_{N_1N_2} - \epsilon_{M_1M_2}\epsilon_{N_1N_2}\right)\right.\\ &\left. + \epsilon^{L_1L_2}\left(\perp_{M_1M_2}\epsilon_{N_1N_2} + \epsilon_{M_1M_2}\perp_{N_1N_2}\right)\right]\\ &\times \frac{\partial^2 \mathcal{L}_E}{\partial R_{M_1R_1N_1S_1}\partial R_{M_2R_2N_2S_2}} K_{L_1R_1S_1} K_{L_2R_2S_2} \,.\end{aligned} \tag{4.70}$$

The operator for the type-A terms (4.50a) becomes:

$$\begin{aligned}\mathcal{K}_{\mathcal{A}I}\hat{\partial}^{\mathcal{A}I} =& \left[\frac{1}{2}\perp^{L_1L_2} h^{P_1P_2}h^{Q_1Q_2}\epsilon_{MN}\epsilon_{RS} - 2\epsilon^{L_1L_2}h^{P_1}_R h^{P_2}_S h^{Q_1Q_2}\epsilon_{MN} - 2\perp^{L_1L_2} h^{P_1}_M h^{Q_1}_R h^{P_2}_N h^{Q_2}_S\right.\\ &\left. - 2\left(\perp^{L_1L_2}\perp_{MR} + \epsilon^{L_1L_2}\epsilon_{MR}\right)h^{P_1}_N h^{P_2}_S h^{Q_1Q_2}\right] K_{L_1P_1Q_1}K_{L_2P_2Q_2}\frac{\partial}{\partial R_{MNRS}}\\ &+ 2\left(\perp_{M_2}{}^{M_1}\perp_{R_2}{}^{R_1} - \epsilon_{M_2}{}^{M_1}\epsilon_{R_2}{}^{R_1}\right) h^{N_1}_{N_2}h^{S_1}_{S_2} R_{M_1N_1R_1S_1}\frac{\partial}{\partial R_{M_2N_2R_2S_2}} \,,\end{aligned} \tag{4.71}$$

while that of type-B terms reads:

$$\mathcal{K}_{\mathcal{B}I}\hat{\partial}^{\mathcal{B}I} = 4\left[\perp^{M_1}_{M_2} h^{N_1}_{N_2}h^{R_1}_{R_2}h^{S_1}_{S_2} + \perp^{M_1}_{M_2}\perp^{N_1}_{N_2}\perp^{R_1}_{R_2} h^{S_1}_{S_2}\right] R_{M_1N_1R_1S_1}\frac{\partial}{\partial R_{M_2N_2R_2S_2}} \,. \tag{4.72}$$

[6]There is an ordering assumption in the value of $\epsilon_{z\bar{z}}$, the normal vectors n^1 and n^2 are defined so that we get $\epsilon_{z\bar{z}} = i/2$

The covariant form of the full holographic entanglement entropy functional can be finally written as:

$$S_{\mathrm{EE}}^{\mathcal{L}_E(\mathrm{Riem})}(A) = -2\pi \int_{\Gamma_A} d^{d-1}y\,\sqrt{h}\left[\epsilon_{MN}\epsilon_{RS}\frac{\partial \mathcal{L}_E}{\partial R_{MNRS}}\right. \tag{4.73}$$

$$\left. -\sum_{S=0}^{\infty}\frac{1}{S!}\int_0^1 \mathrm{d}u\, 2u : \left(-(1-u^2)\mathcal{K}_{\mathcal{A}I}\hat{\partial}^{\mathcal{A}I} - (1-u)\mathcal{K}_{\mathcal{B}J}\hat{\partial}^{\mathcal{B}J}\right)^S : \left(\frac{8\partial^2\mathcal{L}_E}{\partial \mathrm{Riem}^2}K^2\right)\right],$$

where derivatives are to be taken respecting the normal ordering prescription, and the covariant form of the objects appearing in the last line are given in (4.70) – (4.72).

4.3 The HEE functional for some relevant theories

Let us now discuss, using the tools presented in previous sections, the form of the holographic entanglement entropy functional for some particular theories. We will start with a general discussion concerning its structure depending on the number of Riemann tensors present in the Lagrangian; then we will particularize for theories depending only on the Ricci tensor or the Ricci scalar. After this discussion based on the tensors present in the Lagrangian, we look at the important family of Lovelock theories, for which the functional was already known to coincide with the Jacobson-Myers one obtained in the context of black hole entropy [52, 119]. We will recover the same result, but with our proposal for the anomaly part of the entropy written in terms of derivatives. Finally, we give explicit results for quadratic, cubic, and quartic gravities. All results in this section are derived for the perturbative Einstein gravity splitting, (4.39), unless otherwise explicitly stated.

4.3.1 Structure depending on the number of Riemann tensors

Consider the second derivative of the Lagrangian contracted with extrinsic curvatures just like in (4.48), which is the basic object involved in the construction of the anomaly term of the functional. We will show that, unless both derivatives hit Riemann tensors in the Lagrangian (as opposed to Ricci tensors or scalar curvatures), we get a vanishing contribution to the anomaly term. To see this, consider taking one of the derivatives, say, the one with R_{zizj}, and suppose it hits a Ricci tensor, so that we get:

$$\frac{\partial \mathcal{L}_E}{\partial R_{zizj}} = \frac{\partial R_{MN}}{\partial R_{zizj}}T^{MN} + \dots\,, \tag{4.74}$$

where T^{MN} represents the remaining part of the Lagrangian contracted with the Ricci tensor – this can include metric tensors, so the previous expansion is also valid when the derivative hits a Ricci scalar in the Lagrangian. The dots represent the rest of the terms generated by the Leibniz rule. It can be shown from (4.41) that

$$\frac{\partial R_{MN}}{\partial R_{PQRS}} = \delta^{[Q}_{(M}G^{P][R}\delta^{S]}_{N)} \quad \Rightarrow \quad \frac{\partial R_{MN}}{\partial R_{zizj}} = \frac{1}{4}h^{ij}\delta^z_M\delta^z_N\,, \tag{4.75}$$

since $G^{zz} = G^{zi} = 0$. Therefore, the term explicitly written in (4.74) is proportional to h^{ij}. In the anomaly term basic object, (4.48), this is going to be contracted with K_{zij}, so we get:

$$\left(\frac{8\partial^2\mathcal{L}_E}{\partial \mathrm{Riem}^2}K^2\right)=2\frac{\partial T^{MN}}{\partial R_{\bar{z}k\bar{z}l}}K_{zij}K_{\bar{z}kl}h^{ij}\delta^z_M\delta^z_N+\dots=2\frac{\partial T^{zz}}{\partial R_{\bar{z}k\bar{z}l}}K_zK_{\bar{z}kl}+\dots\ , \tag{4.76}$$

which vanishes when evaluated at the RT surface, since $K_z=0$ for a minimal area surface. An analogous argument with the derivative with respect to $R_{\bar{z}k\bar{z}l}$ shows that it does not contribute to the anomaly term at the RT surface if it hits a Ricci tensor. Therefore, only if both derivatives in (4.48) hit uncontracted Riemann tensors in the Lagrangian we can get a non-vanishing contribution.

Consider then an n-th order curvature density containing n_R Riemann tensors and $n-n_R$ Ricci tensors or scalars. After the two derivatives are taken, the only non-vanishing pieces will be of the form

$$\left(\frac{8\partial^2\mathcal{L}_E}{\partial \mathrm{Riem}^2}K^2\right)\sim\sum K^2\mathrm{Ricci}_1\dots\mathrm{Ricci}_{n-n_R}\mathrm{Riem}_1\dots\mathrm{Riem}_{n_R-2}\,. \tag{4.77}$$

We use the symbol $\sim$ to represent the structure of the object in terms of the curvature tensors appearing, ignoring the particular components. The sum means that several terms with this structure will show up in general. Each Ricci_k represents a particular component of the Ricci tensor or scalar and, analogously, Riem_k represents a component of the Riemann tensor. We have to apply now the differential operator in (4.49) to obtain the anomaly term. To do this, let us start by writing explicitly the Ricci tensor and scalar in terms of Riemann tensor components:

$$\begin{aligned} R_{zz}&=h^{ij}R_{zizj}\ , & R_{z\bar{z}}&=-2R_{z\bar{z}z\bar{z}}+h^{ij}R_{zi\bar{z}j}\,,\\ R_{zi}&=-2R_{z\bar{z}zi}+h^{jk}R_{zjik}\,, & R_{ij}&=2R_{zi\bar{z}j}+2R_{zj\bar{z}i}+h^{kl}R_{ikjl}\,,\\ R&=4R_{z\bar{z}}+h^{ij}R_{ij}\,, \end{aligned} \tag{4.78}$$

plus the ones obtained by complex conjugation. Then, the differential operators defined in (4.50) act on these components as follows:

$$\mathcal{K}_{AI}\hat{\partial}^{AI}R_{zz}=R_{zz}\,,\qquad \mathcal{K}_{BI}\hat{\partial}^{BI}R_{zz}=0\,, \tag{4.79a}$$
$$\mathcal{K}_{AI}\hat{\partial}^{AI}R_{z\bar{z}}=0\,,\qquad \mathcal{K}_{BI}\hat{\partial}^{BI}R_{z\bar{z}}=0\,, \tag{4.79b}$$
$$\mathcal{K}_{AI}\hat{\partial}^{AI}R_{zi}=0\,,\qquad \mathcal{K}_{BI}\hat{\partial}^{BI}R_{zi}=R_{zi}\,, \tag{4.79c}$$
$$\mathcal{K}_{AI}\hat{\partial}^{AI}R_{ij}=-K_{aij}K^a\,,\qquad \mathcal{K}_{BI}\hat{\partial}^{BI}R_{ij}=0\,, \tag{4.79d}$$
$$\mathcal{K}_{AI}\hat{\partial}^{AI}R=-K_aK^a\,,\qquad \mathcal{K}_{BI}\hat{\partial}^{BI}R=0\,. \tag{4.79e}$$

Notice also that if the Ricci components are acted upon with several powers of the differential operators in normal order, which is what we must do in order to compute the functional (4.49), the remaining powers would not act on the curvature tensors appearing in the right-hand side of the previous expressions. In any case, the relevant observation is that after applying the differential operator, any Ricci factor in (4.77) generates either something proportional to the very same component or something proportional to K^a. When evaluated at the RT surface, this second possibility gives zero, so in the perturbative functional no Ricci tensor component can ever generate powers of the extrinsic curvature. This is not the case with Riemann tensor components, for which the differential operator generates non-vanishing contractions of extrinsic curvatures in general. The conclusion is that the expression which results from applying the full differential operator of the anomaly term to a second derivative of the form (4.77) has the structure:

$$\begin{aligned}&\sum_\alpha \frac{1}{1+q_\alpha}\left(\frac{8\partial^2\mathcal{L}_E}{\partial \mathrm{Riem}^2}K^2\right)_\alpha \\ &\quad\sim \sum \mathrm{Ricci}^{n-n_R}\left(\mathrm{Riem}^{n_R-2}K^2+\mathrm{Riem}^{n_R-3}K^4+\cdots+\mathrm{Riem}\,K^{2n_R-4}+K^{2n_R-2}\right)\,.\end{aligned} \tag{4.80}$$

When we present the functionals for quadratic, cubic, and quartic theories it will be easy to verify that this structure is respected.

In summary, we can state the previous result as follows: densities containing n_R Riemann curvatures can contain terms involving extrinsic curvatures up to the power $2n_R-2$. In particular, this implies that densities with zero or one Riemann tensors have no anomaly piece. We will study the former case in the following subsection. As for the latter, for a theory (in Euclidean signature) of the form:

$$-\frac{1}{16\pi G_N}\int \mathrm{d}^{d+1}x\sqrt{G}\left[\frac{d(d-1)}{L^2}+R+\lambda R^{MNRS}T_{MNRS}\,(\mathrm{Ricci})\right]\,, \tag{4.81}$$

where $T_{MNRS}\,(\mathrm{Ricci})$ is some tensorial structure involving Ricci tensors and metrics, the corresponding functional comes only from the Wald piece, and it reads:

$$\begin{aligned}\frac{\mathrm{Area}(\Gamma_A)}{4G_N}+\frac{\lambda}{8G_N}\int_{\Gamma_A}\mathrm{d}^{d-1}y\sqrt{h}\Bigg[&2T^{MNRS}\perp_{M[R}\perp_{S]N}\\ &+R^{MNRS}\frac{\partial T_{MNRS}}{\partial R_{PQ}}\perp_{PQ}\Bigg]+\mathcal{O}(\lambda^2)\,.\end{aligned} \tag{4.82}$$

4.3.2 Theories depending on the Ricci tensor

Let us consider now densities constructed from general contractions of the Ricci tensor:

$$I_E^{\mathcal{L}(\mathrm{Ricci})}=-\frac{1}{16\pi G_N}\int \mathrm{d}^{d+1}x\sqrt{G}\left[\frac{d(d-1)}{L^2}+R+\lambda F\left(G_{MN},R_{MN}\right)\right]\,, \tag{4.83}$$

where λ is some coupling constant. The general argument of the previous section shows that, perturbatively, the anomaly term will not contribute to the holographic entanglement entropy functional in these theories: since we only have Ricci tensors (or scalars), the situation in (4.76) is unavoidable. Hence, only the Wald term is relevant and we find:

$$S_{\mathrm{EE}}^{\mathcal{L}(\mathrm{Ricci})}=\frac{\mathrm{Area}(\Gamma_A)}{4G_N}+\frac{\lambda}{8G_N}\int_{\Gamma_A}\mathrm{d}^{d-1}y\sqrt{h}\,\frac{\partial F}{\partial R_{MN}}\perp_{MN}+\mathcal{O}(\lambda^2)\,. \tag{4.84}$$

We emphasize that this formula holds for general-order densities of the form $F\left(G_{MN},R_{MN}\right)$.

4.3.3 $f(R)$ gravities

Consider now theories depending only on the Ricci scalar:

$$I_E^{f(R)} = -\frac{1}{16\pi G_N}\int \mathrm{d}^{d+1}x\sqrt{G}\left[\frac{d(d-1)}{L^2} + R + f(R)\right] . \tag{4.85}$$

These theories are a subclass of the ones considered in the previous subsection, where arbitrary Ricci tensors were allowed. Therefore, we know that they will only contain a Wald-like piece in the holographic entanglement entropy functional, and one might wonder why bother studying them separately. The fundamental difference, already noted in [52], is that:

$$\frac{\partial R}{\partial R_{zizj}} = \frac{\partial R}{\partial R_{\bar{z}k\bar{z}l}} = 0 , \tag{4.86}$$

which implies that not only the anomaly piece is zero perturbatively, this is also true non-perturbatively in the couplings (*i.e.*, we do not need to evaluate in the RT surface to cancel the anomaly term, like we had to do to eliminate (4.76)). The functional is then:

$$S_{\mathrm{EE}}^{f(R)} = \frac{\mathrm{Area}(\Gamma_A)}{4G_N} + \frac{1}{4G_N}\int_{\Gamma_A} \mathrm{d}^{d-1}y\sqrt{h}f'(R) . \tag{4.87}$$

4.3.4 Lovelock gravities

Let us study now the most important higher-curvature generalizations of Einstein gravity: Lovelock theories [118, 121]. These are the most general diffeomorphism invariant, pure-metric theories which have covariantly-conserved, second-order equations of motion. Their Euclidean action is:

$$I_E^{\mathrm{Lovelock}} = -\frac{1}{16\pi G_N}\int \mathrm{d}^{d+1}x\sqrt{G}\left[\frac{d(d-1)}{L^2} + R + \sum_{n=2}^{\lfloor\frac{d+1}{2}\rfloor}\lambda_n L^{2(n-1)}\mathcal{X}_{2n}(R)\right] , \tag{4.88}$$

where $\lfloor x\rfloor$ is the integer part of x, the λ_n are dimensionless couplings and the order-n invariants $\mathcal{X}_{2n}$ are the Lovelock densities:

$$\mathcal{X}_{2n}(R) \equiv \frac{1}{2^n}\delta^{M_1M_2\cdots M_{2n-1}M_{2n}}_{N_1N_2\cdots N_{2n-1}N_{2n}} R^{N_1N_2}{}_{M_1M_2}\cdots R^{N_{2n-1}N_{2n}}{}_{M_{2n-1}M_{2n}} , \tag{4.89}$$

where $\delta^{M_1M_2\cdots M_{2n-1}M_{2n}}_{N_1N_2\cdots N_{2n-1}N_{2n}}$ is the totally antisymmetric product of $2n$ Kronecker deltas. $\mathcal{X}_{2n}$ becomes the Euler density of compact manifolds when evaluated in $2n$ dimensions. The simplest Lovelock theories (besides Einstein gravity) correspond to the Gauss-Bonnet and cubic densities, which read respectively:

$$\mathcal{X}_4 = + R^2 - 4R_{MN}R^{MN} + R_{MNRS}R^{MNRS} , \tag{4.90a}$$

$$\begin{aligned}\mathcal{X}_6 = &+ R^3 - 12R^N_M R^M_N R + 16R^N_M R^R_N R^M_R + 24R_{MNRS}R^{MR}R^{NS}\\ &+ 3RR_{MNRS}R^{MNRS} - 24R_{MNRS}R^{MNR}{}_P R^{SP}\\ &- 8R_M{}^R{}_N{}^S R_R{}^P{}_S{}^Q R_P{}^M{}_Q{}^N + 4R_{MN}{}^{RS}R_{RS}{}^{PQ}R_{PQ}{}^{MN} .\end{aligned} \tag{4.90b}$$

In [52], it was shown using the α expansion of the functional (4.19)–(4.21) that the holographic entanglement entropy is obtained in Lovelock theories by means of the Jacobson-Myers functional, [119]. This had already been guessed in [122] based on consistency with CFT results. We will reach now the same conclusion, but using our derivative expression for the functional, (4.49). This will serve as a further check of its validity. Let us start by observing that, since we have

$$\frac{\partial R^{MN}_{RS}}{\partial R_{zizj}} = \frac{1}{2}\left(G^{z[M}G^{N]i}\delta^{z}_{[R}\delta^{j}_{S]} + G^{z[M}G^{N]j}\delta^{z}_{[R}\delta^{i}_{S]}\right) = 2\delta^{[M}_{\bar{z}}\delta^{N]}_{m}\delta^{z}_{[R}\delta^{(i}_{S]}G^{j)m} \;, \tag{4.91}$$

and a similar result holds for the derivative with respect to $R_{\bar{z}k\bar{z}l}$, the second derivative contracted with K^2 appearing in the anomaly term is of the form:

$$\begin{aligned}\frac{8\partial^2\mathcal{X}_{2n}}{\partial R_{zizj}\partial R_{\bar{z}k\bar{z}l}}K_{zij}K_{\bar{z}kl} = -\frac{8n(n-1)}{2^{n-2}}&\delta^{z\bar{z}ijM_1M_2\ldots M_{2n-5}M_{2n-4}}_{z\bar{z}klN_1N_2\ldots N_{2n-5}N_{2n-4}} \\ &\times R^{N_1N_2}_{M_1M_2}\cdots R^{N_{2n-5}N_{2n-4}}_{M_{2n-5}M_{2n-4}}K_{zi}{}^{k}K_{\bar{z}j}{}^{l} \;.\end{aligned} \tag{4.92}$$

Due to the completely antisymmetric character of the generalized delta, none of the indices M_k or N_k can be z or $\bar{z}$. This forces all components of the Riemann tensor to be of the type $R^{j_1j_2}_{i_1i_2}$:

$$\left(\frac{8\partial^2\mathcal{X}_{2n}}{\partial\mathrm{Riem}^2}K^2\right) = \frac{n(1-n)}{2^{n-3}}\delta^{i_1j_1\ldots i_{n-1}j_{n-1}}_{k_1l_1\ldots k_{n-1}l_{n-1}}R^{k_1l_1}_{i_1j_1}\cdots R^{k_{n-2}l_{n-2}}_{i_{n-2}j_{n-2}}K_{ai_{n-1}}{}^{k_{n-1}}K^{a}{}_{j_{n-1}}{}^{l_{n-1}} \;. \tag{4.93}$$

An important consequence of this is that to compute the anomaly part of the holographic entanglement entropy functional we only have to take into account the part proportional to $\partial/\partial R_{ijkl}$ in (4.50a), because no other Riemann tensor components appear. Therefore, using that:

$$\int_0^1 \mathrm{d}u\, 2u(1-u^2)^S = \frac{1}{S+1} \;, \tag{4.94}$$

the anomaly piece of the entropy (4.49) becomes for a Lovelock density:

$$\begin{aligned}&\sum_{S=0}^{\infty}\frac{1}{(S+1)!}\left(2K_{aik}K^{a}{}_{lj}\frac{\partial}{\partial R_{ijkl}}\right)^{S}\left(\frac{8\partial^2\mathcal{X}_{2n}}{\partial\mathrm{Riem}^2}K^2\right) = \\ &= -\sum_{S=0}^{n-2}\frac{1}{(S+1)!}\frac{n(n-1)\cdots(n-1-S)}{2^{n-3-S}}\delta^{i_1j_1\cdots i_{n-1}j_{n-1}}_{k_1l_1\cdots k_{n-1}l_{n-1}}R^{k_1l_1}_{i_1j_1}\cdots R^{k_{n-2-S}l_{n-2-S}}_{i_{n-2-S}j_{n-2-S}} \\ &\qquad\times K_{a_{n-1-S}i_{n-1-S}}{}^{k_{n-1-S}}K^{a_{n-1-S}}{}_{j_{n-1-S}}{}^{l_{n-1-S}}\ldots K_{a_{n-1}i_{n-1}}{}^{k_{n-1}}K^{a_{n-1}}{}_{j_{n-1}}{}^{l_{n-1}} = \\ &= -n\sum_{S=1}^{n-1}\frac{1}{2^{n-2-S}}\binom{n-1}{S}\delta^{i_1j_1\cdots i_{n-1}j_{n-1}}_{k_1l_1\ldots k_{n-1}l_{n-1}}R^{k_1l_1}_{i_1j_1}\cdots R^{k_{n-1-S}l_{n-1-S}}_{i_{n-1-S}j_{n-1-S}} \\ &\qquad\times K_{a_{n-S}i_{n-S}}{}^{k_{n-S}}K^{a_{n-S}}{}_{j_{n-S}}{}^{l_{n-S}}\cdots K_{a_{n-1}i_{n-1}}{}^{k_{n-1}}K^{a_{n-1}}{}_{j_{n-1}}{}^{l_{n-1}} \;.\end{aligned}$$

Furthermore, the Wald term reads

$$\frac{\partial\mathcal{X}_{2n}}{\partial R_{z\bar{z}z\bar{z}}} = -\frac{n}{2^{n-2}}\delta^{i_1j_1\ldots i_{n-1}j_{n-1}}_{k_1l_1\ldots k_{n-1}l_{n-1}}R^{k_1l_1}_{i_1j_1}\ldots R^{k_{n-1}l_{n-1}}_{i_{n-1}j_{n-1}} \;. \tag{4.95}$$

This can be combined with the previous result for the anomaly, providing the $S=0$ term of the sum. When this is included, the binomial coefficient and the 2^{-S} factor in each term can be employed to write the full functional as:

$$\frac{\partial \mathcal{X}_{2n}}{\partial R_{z\bar{z}z\bar{z}}} + \sum_{S=0}^{\infty} \frac{1}{(S+1)!} \left(2K_{aik} K^{a}{}_{lj} \frac{\partial}{\partial R_{ijkl}} \right)^{S} \left(\frac{8\partial^2 \mathcal{X}_{2n}}{\partial \text{Riem}^2} K^2 \right) =$$
$$= -\frac{n}{2^{n-2}} \delta^{i_1 j_1 \ldots i_{n-1} j_{n-1}}_{k_1 l_1 \ldots k_{n-1} l_{n-1}} \left(R^{k_1 l_1}_{i_1 j_1} + 2K_{a_1 i_1}{}^{k_1} K^{a_1}{}_{j_1}{}^{l_1} \right) \ldots$$
$$\ldots \times \left(R^{k_{n-1} l_{n-1}}_{i_{n-1} j_{n-1}} + 2K_{a_{n-1} i_{n-1}}{}^{k_{n-1}} K^{a_{n-1}}{}_{j_{n-1}}{}^{l_{n-1}} \right) , \tag{4.96}$$

where we used the fact that the binomial factor is the number of ways we can pick S squared extrinsic curvature factors and $(n-1-S)$ Riemann tensors from the previous product (and the antisymmetric delta can be used to rewrite all possible combinations as essentially the same). The final observation is that, due to the antisymmetric properties of the generalized δ and the splitting (4.24), the combination in each bracket is $\tilde{R}^{kl}_{ij}$. This, as shown in [52], is nothing but the intrinsic curvature tensor of the surface Γ_A, which we denote $\mathcal{R}^{kl}_{ij}$. It follows that we can write the holographic entanglement entropy functional for a given order-n Lovelock density as

$$S_{\text{EE}}^{\mathcal{X}_{2n}} = -4\pi n \int_{\Gamma_A} \mathrm{d}^{d-1} y \sqrt{h}\, \mathcal{X}_{2(n-1)}(\mathcal{R}) , \tag{4.97}$$

which is the Jacobson-Myers form, [119]. This has the interesting property of being fully determined in terms of intrinsic curvatures associated to the holographic entangling surface.

We can go back now to our general theory containing a linear combination of Lovelock densities, (4.88). Using the previous result, we conclude that the holographic entanglement entropy functional is:

$$S_{\text{EE}}^{\text{Lovelock}} = \frac{\text{Area}(\Gamma_A)}{4G_N} + \sum_{n=2}^{\lfloor \frac{d+1}{2} \rfloor} \frac{n L^{2(n-1)}}{4G_N} \int_{\Gamma_A} \mathrm{d}^{d-1} y \sqrt{h} \lambda_n \mathcal{X}_{2(n-1)}(\mathcal{R}) , \tag{4.98}$$

where the lower-order densities are computed with respect to the induced metric h_{ij}.

4.3.5 Quadratic, cubic, and quartic gravities

We conclude this chapter with a brief discussion of the holographic entanglement entropy functional obtained for higher-curvature theories containing up to four powers of the curvature tensors. The simplest case is provided by quadratic gravity:

$$I_E^{\text{Riem}^2} = -\frac{1}{16\pi G_N} \int \mathrm{d}^{d+1} x \sqrt{G} \left[\frac{d(d-1)}{L^2} + R + L^2 \sum_{i=1}^{3} \alpha_i \mathcal{L}_i^{(2)} \right] , \tag{4.99}$$

where

$$\mathcal{L}_1^{(2)} \equiv R^2 , \quad \mathcal{L}_2^{(2)} \equiv R_{MN} R^{MN} , \quad \mathcal{L}_3^{(2)} \equiv R_{MNRS} R^{MNRS} . \tag{4.100}$$

The holographic entanglement entropy functional for this class of theories was first obtained in [113]. It reads:

$$S_{\rm EE}^{\rm Riem^2}=\frac{\text{Area}(\Gamma_A)}{4G_N}+\frac{L^2}{4G_N}\int_{\Gamma_A}\mathrm{d}^{d-1}y\sqrt{h}\sum_{i=1}^{3}\alpha_i\Delta_i^{(2)}\,, \tag{4.101}$$

where

$$\Delta_1^{(2)}=2R\,,\quad \Delta_2^{(2)}=R_a^a-\frac{1}{2}K^aK_a\,,\quad \Delta_3^{(2)}=2\left(R_{ab}^{ab}-K_{aij}K^{aij}\right)\,. \tag{4.102}$$

Just like for $f(R)$ and Lovelock theories, there is no splitting problem in this case. This happens because after two derivatives of the Lagrangian we have no curvature tensors left. Thus, splitting ambiguities do not arise, and (4.101) can be trusted at all orders in α_i. When the terms are considered as perturbative corrections to Einstein gravity, the above expressions get slightly simplified, namely:

$$S_{\rm EE}^{\rm Riem^2}=\frac{\text{Area}(\Gamma_A)}{4G_N}+\frac{L^2}{4G_N}\int_{\Gamma_A}\mathrm{d}^{d-1}y\sqrt{h}\sum_{i=1}^{3}\alpha_i\Delta_i^{(2)}+\mathcal{O}(\alpha_i^2)\,, \tag{4.103}$$

where now

$$\Delta_1^{(2)}=2R\,,\quad \Delta_2^{(2)}=R_a^a\,,\quad \Delta_3^{(2)}=2\left(R_{ab}^{ab}-K_{aij}K^{aij}\right)\,. \tag{4.104}$$

Similarly, we can consider the most general cubic theory containing only contractions of the curvature tensors. There are eight independent invariants:

$$I_E^{\rm Riem^3}=-\frac{1}{16\pi G_N}\int\mathrm{d}^{d+1}x\sqrt{G}\left[\frac{d(d-1)}{L^2}+R+L^4\sum_{i=1}^{8}\beta_i\mathcal{L}_i^{(3)}\right]\,. \tag{4.105}$$

We label our basis of densities as follows:

$$\mathcal{L}_1^{(3)}\equiv R_{M\ N}^{\ R\ S}R_{R\ S}^{\ P\ Q}R_{P\ Q}^{\ M\ N}\,,\qquad \mathcal{L}_2^{(3)}\equiv R_{MN}^{\ \ \ RS}R_{RS}^{\ \ \ PQ}R_{PQ}^{\ \ \ MN}\,, \tag{4.106a}$$

$$\mathcal{L}_3^{(3)}\equiv R_{MNRS}R^{MNR}{}_{P}R^{SP}\,,\qquad \mathcal{L}_4^{(3)}\equiv R_{MNRS}R^{MNRS}R\,, \tag{4.106b}$$

$$\mathcal{L}_5^{(3)}\equiv R_{MNRS}R^{MR}R^{NS}\,,\qquad \mathcal{L}_6^{(3)}\equiv R_M^NR_N^RR_R^M\,, \tag{4.106c}$$

$$\mathcal{L}_7^{(3)}\equiv R_{MN}R^{MN}R\,,\qquad \mathcal{L}_8^{(3)}\equiv R^3\,. \tag{4.106d}$$

The corresponding functional has the form:

$$S_{\rm EE}^{\rm Riem^3}=\frac{\text{Area}(\Gamma_A)}{4G_N}+\frac{L^4}{4G_N}\int_{\Gamma_A}\mathrm{d}^{d-1}y\sqrt{h}\sum_{i=1}^{8}\beta_i\Delta_i^{(3)}+\mathcal{O}(\beta_i^2)\,. \tag{4.107}$$

The particular expressions for $\Delta_i^{(3)}$ are quite complicated, and not particularly illuminating. For this reason, we relegate them to appendix C. Let us only emphasize that their structure coincides with the one expected based on the arguments that led to (4.80). All terms have a Wald piece which is quadratic in curvatures, and densities with no Riemann tensors or with a single one do not present any extra terms coming from the anomaly. Densities with two Riemann tensors have terms of the form K^2R (where K is the extrinsic curvature and R a certain bulk curvature tensor), and those with three Riemann tensors present in addition terms of the form K^4.

Finally, we consider the most general quartic theory without derivatives of the curvature tensors. There are 26 independent densities – see *e.g.*, [123, 124]:

$$I_E^{\mathrm{Riem}^4} = -\frac{1}{16\pi G_N}\int \mathrm{d}^{d+1}x\sqrt{G}\left[\frac{d(d-1)}{L^2} + R + L^6\sum_{i=1}^{26}\gamma_i\mathcal{L}_i^{(4)}\right] , \tag{4.108}$$

where we choose our basis to be

$$\mathcal{L}_{26}^{(4)} \equiv R^4 , \qquad \mathcal{L}_{25}^{(4)} \equiv R^2 R_{MN}R^{MN} , \tag{4.109a}$$

$$\mathcal{L}_{24}^{(4)} \equiv R R_M{}^N R_N{}^R R_R{}^M , \qquad \mathcal{L}_{23}^{(4)} \equiv R_{MN}R^{MN}R_{RS}R^{RS} , \tag{4.109b}$$

$$\mathcal{L}_{22}^{(4)} \equiv R_M{}^N R_N{}^R R_R{}^S R_S{}^M , \qquad \mathcal{L}_{21}^{(4)} \equiv R R_{MNRS}R^{MR}R^{NS} , \tag{4.109c}$$

$$\mathcal{L}_{20}^{(4)} \equiv R^{MN}R_{MRNS}R^{PR}R_P{}^S , \qquad \mathcal{L}_{19}^{(4)} \equiv R^2 R_{MNRS}R^{MNRS} , \tag{4.109d}$$

$$\mathcal{L}_{18}^{(4)} \equiv R R_{MNRS}R^{MNR}{}_P R^{SP} , \qquad \mathcal{L}_{17}^{(4)} \equiv R_{PQ}R^{PQ}R_{MNRS}R^{MNRS} , \tag{4.109e}$$

$$\mathcal{L}_{16}^{(4)} \equiv R^{MN}R_N{}^R R^{SPQ}{}_M R_{SPQR} , \qquad \mathcal{L}_{15}^{(4)} \equiv R^{MN}R^{RS}R^{PQ}{}_{MR}R_{PQNS} , \tag{4.109f}$$

$$\mathcal{L}_{14}^{(4)} \equiv R^{MN}R^{RS}R^P{}_M{}^Q{}_N R_{PRQS} , \qquad \mathcal{L}_{13}^{(4)} \equiv R^{MN}R^{RS}R^P{}_M{}^Q{}_R R_{PNQS} , \tag{4.109g}$$

$$\mathcal{L}_{12}^{(4)} \equiv R R_{MN}{}^{RS}R_{RS}{}^{PQ}R_{PQ}{}^{MN} , \qquad \mathcal{L}_{11}^{(4)} \equiv R R_M{}^R{}_N{}^S R_R{}^P{}_S{}^Q R_P{}^M{}_Q{}^N , \tag{4.109h}$$

$$\mathcal{L}_{10}^{(4)} \equiv R^{MN}R_M{}^R{}_N{}^S R_{PQUR}R^{PQU}{}_S , \qquad \mathcal{L}_{9}^{(4)} \equiv R^{MN}R^{RSPQ}R_{RS}{}^U{}_M R_{PQUN} , \tag{4.109i}$$

$$\mathcal{L}_{8}^{(4)} \equiv R^{MN}R^{RSPQ}R_R{}^U{}_{PM}R_{SUQN} , \qquad \mathcal{L}_{7}^{(4)} \equiv R_{MNRS}R^{MNRS}R_{PQUT}R^{PQUT} , \tag{4.109j}$$

$$\mathcal{L}_{6}^{(4)} \equiv R^{MNRS}R_{MNR}{}^P R_{QUTS}R^{QUT}{}_P , \qquad \mathcal{L}_{5}^{(4)} \equiv R^{MNRS}R_{MN}{}^{PQ}R_{PQ}{}^{TU}R_{RSTU} , \tag{4.109k}$$

$$\mathcal{L}_{4}^{(4)} \equiv R^{MNRS}R_{MN}{}^{PQ}R_{RP}{}^{TU}R_{SQTU} , \qquad \mathcal{L}_{3}^{(4)} \equiv R^{MNRS}R_{MN}{}^{PQ}R_R{}^T{}_P{}^U R_{STQU} , \tag{4.109l}$$

$$\mathcal{L}_{2}^{(4)} \equiv R^{MNRS}R_M{}^P{}_R{}^Q R_P{}^T{}_Q{}^U R_{NTSU} , \qquad \mathcal{L}_{1}^{(4)} \equiv R^{MNRS}R_M{}^P{}_R{}^Q R_P{}^T{}_N{}^U R_{QTSU} . \tag{4.109m}$$

The entanglement entropy functional has the form:

$$S_{\mathrm{EE}}^{\mathrm{Riem}^4} = \frac{\mathrm{Area}(\Gamma_A)}{4G_N} + \frac{L^6}{4G_N}\int_{\Gamma_A}\mathrm{d}^{d-1}y\sqrt{h}\sum_{i=1}^{26}\gamma_i\Delta_i^{(4)} + \mathcal{O}(\gamma_i^2) , \tag{4.110}$$

where, as in the cubic case, the $\Delta_i^{(4)}$ are presented in appendix C because they are too messy to be presented here. Let us only remark that, to obtain these contributions to the holographic entanglement entropy, it was essential the expression in terms of derivatives (4.49), since it was possible to implement it in the tensor computer algebra package *xAct* [55]. Without this aid, obtaining the quartic functionals by means of the α expansion would be certainly out of reach.

4.4 Final discussion and conclusions

We started the present chapter with a discussion on holographic entanglement entropy in Einstein gravity, for which the Ryu-Takayanagi proposal is available. As shown there, when higher-curvature terms are included in the bulk, the corresponding recipe to holographically find the entanglement entropy of a given boundary region becomes increasingly involved. On the one hand, one has to face the splitting problem, which for a general theory with non-perturbative higher-curvature couplings implies we must consider the full bulk equations of motion in the regularized geometry needed to perform the replica trick which leads to the holographic entanglement entropy functional. This, to the best of our knowledge, has not been explicitly done in any non-trivial higher-curvature gravity.[7] On the other hand, even if one gets rid of the splitting problem by treating the couplings perturbatively, the procedure to find the functional is cumbersome, and in particular it involves an obscure weighted α-sum badly suited for both computations and conceptual discussions.

The contribution of the present chapter aims at improving this situation, working always perturbatively in the couplings to avoid the difficulties of the splitting problem. The α-sum can be conveniently rewritten in terms of differential operators, way easier to be dealt with, as presented in (4.49) – or, more compactly, in (4.52). This has several obvious advantages, the first one being the fact that understanding what the anomaly term adds to the functional is clearer now. The differential operators in (4.49) simply generate higher-order terms in extrinsic curvatures, in a different form depending on the normal or tangential character of curvature components with respect to the bulk surface. Furthermore, the expression is easier to implement computationally, and in particular we have been able to use it to obtain the holographic entanglement entropy functional for theories containing up to quartic curvature corrections. If desired, considering higher orders should also be possible with the help of an efficient implementation of our formula in some appropriate mathematical software. Doing something analogous with the α-sum would have been painful, at the very least.

The advantages of the rewriting of the holographic entanglement entropy functional are not only computational, however. We also gain on the conceptual side, as the general results obtained depending on the structure of the theory show. Furthermore, even if the new expression is not directly applicable outside the perturbative regime due to the splitting problem, we expect its underlying philosophy to be applicable in case a different splitting emerges from considerations of the full bulk equations of motion for a certain higher-curvature theory. The derivation has been done starting from the general form (4.40), where each Riemann tensor component provides a $\tilde{R}_{\mathcal{M}I}$ which does not contribute to q_α in the α-sum, and a $\mathcal{K}_{\mathcal{M}I}$ with a definite non-zero value of q_α. We considered two kinds of contributions: those with $q_\alpha = 1$ (type-A terms), and those with $q_\alpha = 1/2$ (type-B terms). If a different splitting implies new types of weights, the subsequent steps towards the formula in terms of differential operators should be easy to adapt.

Finally, the new version of the holographic entanglement entropy might be useful for some interesting speculations on black hole entropy in higher-curvature gravities. In particular, given that the same functional evaluated on the horizon should be used to compute black hole entropy – this is a consequence of the duality between black holes and thermal states in the field theory –, we can ask ourselves whether it is possible to use it to show the conditions under which a second law is valid. The extrinsic curvature terms are key for this, since a non-stationary black hole will have contributions from them. These are not captured by Wald's piece of the entropy, which is only valid in a stationary

[7] There are good reasons for this, as the equations of motion are generically complicated. Recall that certain theories, such as $f(R)$, quadratic, and Lovelock gravities do not suffer from the splitting problem. Therefore, in these cases it is known how to work non-perturbatively in the couplings.

situation. The exponential rewriting of the functional, (4.52), seems particularly well-suited for considering higher-curvature theories of a general order. Therefore, it would be interesting to see if it can add anything new to the results obtained in previous works on the second law for higher-curvature gravities, such as [125–127]. We will however leave such spectulations as open questions for future work, and in the next chapter we will put to good use the holographic entanglement entropy functionals obtained for cubic theories by computing certain interesting CFT quantities.

5 Universal terms of entanglement entropy

Entanglement entropy in field theories is generically a UV-divergent quantity. This should come as no surprise: UV-divergences are ubiquituous in quantum field theory, and a quantity so closely related to the number of degrees of freedom of a system as the entanglement entropy is certainly a candidate to behave badly when taking the continuum limit, which implies going to an infinite number of degrees of freedom. This non-rigorous notion can be made much more precise, to the point it is possible to show that smoothness of a given quantum state demands a particular entanglement structure between nearby points [128]. However, the divergent character of the entanglement entropy is far from making it a useless concept. First of all, the divergences are somewhat under control. We will elaborate more on this below, but let us just mention now the archetypical example: the entanglement entropy in $d > 2$ dimensions is known to satisfy an *area law* for the leading divergence. This was noted in [129, 130], where it was shown that for a massless scalar field in d-dimensional flat spacetime (with $d > 2$), the entanglement entropy of a region A of typical size ℓ in the global vacuum state of the whole system is given by:

$$S_{\mathrm{EE}} \sim \frac{\mathrm{Area}(\partial A)}{\delta^{d-2}} + \ldots \sim \frac{\ell^{d-2}}{\delta^{d-2}} + \ldots \ , \tag{5.1}$$

where ∂A denotes the boundary of A, and δ is a UV-regulator with dimensions of length (which can be thought of as a lattice spacing in a regularized version of the field theory, with $\delta \to 0$ being the continuum limit). This triggered many speculations at the time about the possibility that black hole entropy, which is also proportional to the area, could be in some sense attributed to entanglement entropy of the fields living outside the horizon [131]. We refer the reader to [132] for a nice review of different works related to this idea.

Even when UV-divergences are not directly tackled with a regularization procedure, entanglement entropy has proved to be an extremely useful concept to improve our theoretical understanding of field theories. This is nicely exemplified by the so-called entropic $\mathcal{C}$-theorems, which are very general results in the context of Wilsonian Renormalization Group (RG) flows. Loosely speaking, these theorems formalize the intuitive idea that the RG flow proceeds by coarse graining the microscopic degrees of freedom, so that there must be a function which is directly related to the number of degrees of freedom and which monotonically decreases along the flow. This is called a $\mathcal{C}$-function, and by the previous argument $\mathcal{C}_{\mathrm{UV}} \geq \mathcal{C}_{\mathrm{IR}}$, $\mathcal{C}_{\mathrm{UV}}$ and $\mathcal{C}_{\mathrm{IR}}$ being the values at the UV and IR fixed points of a given flow. The first example of a $\mathcal{C}$-function made no reference to entanglement entropy: it was obtained by Zamolodchikov [133] for two-dimensional renormalizable quantum field theories, and for historical reasons in this dimensionality the result is called the c-theorem. Remarkably, a different c-function for two-dimensional field theories was obtained by Casini and Huerta [134] from the entanglement entropy of an interval of length ℓ, $S_{\rm EE}(\ell)$:

$$c(\ell) = 3\ell\, S'_{\rm EE}(\ell)\ , \tag{5.2}$$

where prime denotes derivative with ℓ. Using the strong subadditivity property of the entanglement entropy as well as Lorentz invariance, [134] shows that $c'(\ell) \leq 0$. Thus, this is a non-increasing function when going to larger distances, which is the direction of the RG flow.[1] Strikingly enough, the strategy of looking for $\mathcal{C}$-functions defined by means of entanglement entropy has proved to be extremely fruitful. In 3-dimensional field theories, where the proposed $\mathcal{C}$-function is known as an F-function, the only available proof of its monotonicity along the RG flow is obtained by exploiting properties of the entanglement entropy, and therefore is known as the entropic F-theorem [135]. Even the 4-dimensional analogous a-theorem, which is the highest dimension for which a $\mathcal{C}$-theorem is available, can be proven using entropic techniques [136], although in this case another approaches are also available [137]. $\mathcal{C}$-theorems are a broad field of research, a nice place to get an overview is [138]. In any case, entanglement entropy and techniques based on it have been fundamental in advancing this program, showing that, despite divergences, the physical content of the entanglement entropy of field theory regions is rich and interesting.

In this chapter, we intend to extract some of this physically relevant information entanglement entropy contains, but making use of a UV-regulator and isolating pieces which do not depend on the particular choice of regularization scheme. To understand how this works, let us consider a CFT (in which we do not have any dimensionful parameter at hand), and employ the replica trick presented in the previous chapter to write the entanglement entropy of a certain region as in (4.7):

$$S_{\rm EE}(A) = -\,\partial_n \left(\log \mathcal{Z}_n - n \log \mathcal{Z}\right)\big|_{n=1}\ . \tag{5.3}$$

with $\mathcal{Z}_n$ the n-fold cover. The partition function in the absence of dimensionful parameters will have the following UV-divergent structure:

$$\log \mathcal{Z}_n = \sum_{k=0}^{\lfloor d/2 \rfloor} C_{d-2k} \int_{\mathcal{M}_n} \mathrm{d}^d x\, \sqrt{g}\, \Lambda^{d-2k} R^k + (-1)^{(d-1)/2} F\ , \tag{5.4}$$

where Λ is a UV energy cutoff, R^k schematically denotes some polynomial of order k of curvature tensors of the n-fold cover, and F is some finite piece in the previous expansion. This expression comes only from power-counting the possible divergences in a covariant conformal theory. There are a couple of cases which will be relevant for us and in which some extra terms must be added: the conformal anomaly in even-dimensional spacetimes can introduce a logarithmic divergence in the previous expression, while considering non-smooth regions in the definition of the n-fold cover can produce boundary terms with a

[1] Incidentally, at a fixed point of the RG flow, where the theory is conformal, c coincides with the central charge of the CFT. This can be shown from the general form of the entropy for (1+1)-dimensional CFTs, discussed in (5.7).

different UV-divergent structure. For the sake of simplicity, let us ignore the possibility of boundary terms momentarily. Using (5.4) in (5.3), we see that the leading volume divergence cancels between the two terms because it contains no curvature terms. The remaining ones will produce, schematically:

$$S_{\mathrm{EE}}(A) = \frac{c_{d-2}}{\delta^{d-2}} + \frac{c_{d-4}}{\delta^{d-4}} + \ldots + \begin{cases} \dfrac{c_2}{\delta^2} + c_0 \log\left(\dfrac{\ell}{\delta}\right) + (-1)^{(d-1)/2} F + \ldots & d \text{ even} \\ \dfrac{c_1}{\delta} + (-1)^{(d-1)/2} F + \ldots & d \text{ odd} \end{cases}, \tag{5.5}$$

where c_{d-2k} are local integrals on the boundary ∂A of the region considered, and we traded the UV energy regulator for a small length cutoff, $\delta \sim \Lambda^{-1}$. In the logarithmic term, ℓ denotes a length scale characterizing A. Notice that the dominant divergence is the area-law one, so this generalizes (5.1) – although it shows that, in $d = 2$, this area-law divergence does not appear and the logarithmic term is the dominant one, which incidentally makes the entropic c-function (5.2) well-defined. Most of the quantities appearing in the previous expression are regulator-dependent: changing the value of δ changes their value. There are, however, two parameters which are unambiguously defined: these are the so-called universal terms of the entanglement entropy, and they contain physically meaningful information about the CFT.

In even dimensions, c_0 is not affected by a redefinition of the cutoff, for it appears multiplying the logarithmic divergence. Since we argued that this piece comes from the conformal anomaly, it is no surprise c_0 is actually related to the coefficients appearing in this conformal anomaly. Generically, these are [139, 140]:

$$\langle T^\mu_\mu \rangle = 2A\mathcal{X}_d + \sum_i B_i I_i + \nabla_\mu J^\mu, \tag{5.6}$$

where $\mathcal{X}_d$ is the d-dimensional Euler density, I_i are a set of invariants built out from the d-dimensional Weyl tensor, and $\nabla_\mu J^\mu$ is scheme dependent and can be modified by means of local counterterms in the effective action, so it will not show up in physical results. The coefficients A and B_i are known as type-A and type-B anomalies. In two dimensions there are no Weyl invariants, so only the type-A anomaly is present and the coefficient is usually called c (with some extra numerical factors). All in all, in the entanglement entropy it appears as [138, 141]:

$$S^{2d}_{\mathrm{EE}}(A) = \frac{c}{3} \log\left(\frac{\ell}{\delta}\right) + \ldots\ , \tag{5.7}$$

where ℓ in this case is the length of a one-dimensional interval A. In four dimensions, the trace anomaly has contributions both from the Euler density and the Weyl term. It is conventionally written as:

$$\langle T^\mu_\mu \rangle = -\frac{a}{16\pi^2}\mathcal{X}_4 + \frac{c}{16\pi^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}\ , \tag{5.8}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor of the curved manifold in which the CFT is considered, and $\mathcal{X}_4 \equiv R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\mu\nu} - 4R^\mu_\nu R^\nu_\mu + R^2$. The universal contribution c_0 to the entanglement entropy for a smooth entangling surface is given by Solodukhin's formula [113, 142]:

$$c_0^{4d} = \frac{1}{2\pi} \int_{\partial A} \left[-a\mathcal{R} + c\left(R^{\mu\nu}_{\rho\sigma} \perp^\rho_\mu \perp^\sigma_\nu - R^\mu_\nu \perp^\nu_\mu + \frac{1}{3} R + \frac{1}{2} k_a k^a - k_{a\mu\nu} k^{a\mu\nu} \right) \right]\ , \tag{5.9}$$

where $\mathcal{R}$ is the intrinsic Ricci scalar of ∂A, $\perp_{\mu\nu}$ is the metric in the orthogonal subspace to ∂A, $k_{a\mu\nu}$ are the extrinsic curvatures of ∂A, and k_a is the trace $k_{a\mu}{}^{\mu}$. Notice that all these quantities are given by the geometry in which the field theory lives, we have not yet mentioned holography or a dual bulk spacetime. Index a runs over the two normal directions to ∂A (which is 2-dimensional) in the 4-dimensional geometry. Furthermore, if this background space is flat, only the terms proportional to $\mathcal{R}$ and those with extrinsic curvatures survive, since the curvature tensors of the full manifold vanish. The integral of $\mathcal{R}$ in the 2-dimensional surface ∂A is its Euler characteristic (a topological invariant), which is zero for a cylinder and 2 for a sphere. On the contrary, extrinsic curvatures $k_{a\mu\nu}$ for a sphere embedded in flat space satisfy the relation $\frac{1}{2}k_a k^a - k_{a\mu\nu}k^{a\mu\nu} = 0$, while this combination is non-vanishing for a cylinder. Therefore, we can isolate coefficient a by computing the universal term of the entanglement entropy for a spherical region, while coefficient c is obtained using a cylindrical region instead. We will not continue increasing the dimension at this point, but a similar game can be played for larger, even d. The $d = 6$ case will be relevant later on, but we postpone its discussion until the specific calculations are presented.

In odd dimensions the story is slightly different, because we do not have conformal anomaly. No logarithmic divergence appears in (5.5), and the universal term is F, which does not change under cutoff redefinition. Is it possible to interpret F as a known CFT quantity? At least if we consider A to be a spherical entangling region in flat space, the answer is yes. By means of a cleverly chosen conformal transformation [143, 144], we can map flat space $\mathbb{R}^d$ to a Euclidean hyperbolic space $\mathbb{S}^1 \times \mathbb{H}^{d-1}$ at temperature $1/(2\pi)$; under this transformation one can show that $(-1)^{(d-1)/2}F$ is the corresponding thermal entropy on $\mathbb{H}^{d-1}$. A further conformal transformation takes us to $\mathbb{S}^d$, in which case F is identified with the logarithm of the partition function in the Euclidean sphere, $F = (-1)^{(d-1)/2} \log \mathcal{Z}[\mathbb{S}^d]$. Thus, for a spherical entangling region A in flat space we have a clear notion of the meaning of the universal term in odd dimensions.

All in all, the preceding discussion shows that computing the regularized entanglement entropy of a region as in (5.5) and then extracting the universal piece can give us a lot of physically relevant information about a given CFT. Since AdS/CFT provides a simple way to compute entanglement entropies of many regions – generically much easier than field theory computations, when these are available –, it is interesting to study different boundary regions in an holographic context, extracting in each case the universal pieces. This was done from the very beginning after the Ryu-Takayanagi proposal for Einstein gravity appeared [33, 105]. If we restrict ourselves to Einstein gravity, the class of boundary CFTs we can study is limited: as a simple example, for Einstein gravity in 5 dimensions the dual conformal field theory has equal trace anomaly coefficients, $a = c$. We can explore a broader set of theories if we include bulk higher-curvature corrections; this strategy has also been pursued in the past [54, 116, 122, 145, 146]. Our goal now will be to perform a similar analysis, but making use of the results obtained in the previous chapter, which allow us to consider the most general theory including up to cubic Riemann perturbative corrections to Einstein gravity in the bulk.[2] We follow the presentation of [5], where entanglement entropy was obtained for different regions: spheres, slabs, cylinders,

[2]Quartic corrections can be included also based on our previous results, but this is probably too much of an extra complication for little profit. The new phenomenology provided by cubic theories, especially regarding regions with corners, will be enough.

and corners. These last singular regions will turn out to be particularly interesting. On the one hand, the non-smooth character of the corner produces a modification of (5.5), but we can nevertheless identify a universal term which is cutoff independent and therefore physically relevant. On the other, bulk corrections cubic in curvatures will allow to obtain a result for this universal term different from the Einstein gravity one, something which does not happen when including quadratic corrections [54]. This is therefore the first holographic corner function which differs from the Einstein gravity one. In what follows, we restrict ourselves to the dual CFT vacuum state in flat space, which means we work in Poincaré AdS_{d+1} in the bulk.

5.1 Spherical regions and some general results

We will start by considering the entanglement entropy of a boundary ball, A. This means $\partial A = \mathbb{S}^{d-2}$ for a d-dimensional theory. Before computing the detailed form of the holographic entanglement entropy, let us mention some general consequences of working in the vacuum state, which guarantees our bulk dual to be AdS_{d+1}. This is a maximally symmetric space, which necessarily implies:

$$R^{MN}_{RS}\big|_{\mathrm{AdS}} = -\frac{2}{L_\star}\delta^{M}_{[R}\delta^{N}_{S]}\ , \qquad \left.\frac{\partial\mathcal{L}_E}{\partial R_{MNRS}}\right|_{\mathrm{AdS}} = 2k_0 G^{M[R}G^{S]N}\ , \tag{5.10}$$

where the second equation follows from symmetry properties and the fact that all curvature tensors reduce to metrics in a maximally symmetric background. The curvature scale of AdS, $L_\star$, can be obtained in terms of the couplings by plugging the previous Riemann tensor in the vacuum equations of motion of the particular theory and solving the resulting polynomial equation. This can be shown to be equivalent to the following condition [124]:

$$k_0 = -\frac{L_\star^2}{4d}\,\mathcal{L}_E|_{\mathrm{AdS}}\ . \tag{5.11}$$

We work in Euclidean signature to connect with results in the previous chapter, but identical results exist in Lorentzian signature. The first derivative of the Euclidean Lagrangian in the background AdS geometry is precisely the quantity appearing in the Wald term of the holographic entanglement entropy functional, (4.20). This can be rewritten as:

$$S_{\mathrm{Wald}}|_{\mathrm{AdS}} = -8\pi k_0\int_{\Gamma_A}\mathrm{d}^{d-1}y\,\sqrt{h} = -32\pi G_N k_0 S^{\mathrm{E}}_{\mathrm{EE}}\ . \tag{5.12}$$

This shows that the Wald pieces will always be proportional to the Einstein gravity result for the entanglement entropy. k_0 can be related to the universal term for a spherical surface, so we turn now to showing this.

Write the Poincaré AdS_{d+1} metric as

$$\mathrm{d}s^2 = \frac{L_\star^2}{z^2}\left[\mathrm{d}\tau^2+\mathrm{d}z^2+\mathrm{d}r^2+r^2\mathrm{d}\Omega^2_{d-2}\right]\ , \tag{5.13}$$

where $\mathrm{d}\Omega^2_{d-2}$ is the metric of the usual round sphere. Our entangling surface is a boundary sphere (at fixed τ and $z\to 0$) of radius $r=\ell$ centered at $r=0$. As argued in the previous chapter, for perturbative corrections to Einstein gravity in the bulk we have to consider the minimal area RT surface to evaluate the holographic entanglement entropy. Let us exploit the symmetry of the problem to parametrize it as $\tau = 0$, $z = Z(r)$. Then, bulk unit normals to the surface are given by:

$$n_1 = \frac{z}{L_\star}\partial_\tau \ , \qquad n_2 = \frac{z}{L_\star\sqrt{1+Z'^2}}\left(Z'\partial_r - \partial_z\right) \ . \tag{5.14}$$

We have already extended these vector fields to a neighborhood of the surface while keeping them normalized. On the surface, one fixes $z = Z(r)$, and $Z'(r)$ is well-defined for any (r, z) with $r \in (0, \ell)$. The induced metric on the surface is given by

$$h_{MN}\mathrm{d}x^M\mathrm{d}x^N = \frac{L_\star^2}{Z^2}\left[\frac{1}{1+Z'^2}\left(\mathrm{d}r + Z'\mathrm{d}z\right)^2 + r^2\mathrm{d}\Omega_{d-2}^2\right] . \tag{5.15}$$

With these results one can compute in full generality the components of the extrinsic curvatures,

$$K^1{}_{MN} = 0 \ , \tag{5.16a}$$

$$K^2{}_{rr} = \frac{L_\star}{Z^2(1+Z'^2)^{5/2}}\left(1 + Z'^2 + ZZ''\right) \ , \quad K^2{}_{rz} = Z'K^2{}_{rr} \ , \tag{5.16b}$$

$$K^2{}_{zz} = Z'^2K^2{}_{rr} \ , \quad K^2{}_{mn} = \frac{L_\star}{Z^2\sqrt{1+Z'^2}}\left(ZZ' + r\right) r\hat{g}_{mn} \ , \tag{5.16c}$$

where $\hat{g}_{mn}$ is the metric of the unit $\mathbb{S}^{d-2}$. Obtaining the traces is now immediate: $K^1 = 0$ trivially, whereas

$$K^2 = \frac{1}{L_\star r(1+Z'^2)^{3/2}}\left[rZZ'' + (d-2)ZZ'(1+Z'^2) + (d-1)r(1+Z'^2)\right] . \tag{5.17}$$

The vanishing of this trace is exactly the differential equation for the surface one would obtain by minimizing the RT functional, which in this case reads, after integrating in Ω_{d-2}:

$$S_{\mathrm{EE}}^{\mathrm{E}} = \frac{L_\star^{d-1}\pi^{(d-1)/2}}{2G_N\Gamma\left[\frac{d-1}{2}\right]}\int_0^\ell \mathrm{d}r\, \frac{r^{d-2}}{Z^{d-1}}\sqrt{1+Z'^2} \, . \tag{5.18}$$

The solution of the differential equation $K^2 = 0$ satisfying the boundary condition $Z(\ell) = 0$ is $r^2 + Z^2 = \ell^2$. The simplicity of this RT surface has another important consequence: since $ZZ' = -r$ and $ZZ'' = -(1+Z'^2)$, the extrinsic curvature $K^2{}_{MN}$ vanishes. Thus, both $K^1{}_{MN}$ and $K^2{}_{MN}$ are zero for the RT surface. Since the anomaly term in the general higher-curvature functional is quadratic in extrinsic curvatures of the surface, it will not contribute for spherical regions and we just have to consider the Wald piece, (5.12).[3] Being it proportional to the Einstein gravity result, it will prove useful to compute this first. Let us introduce a regulator at $z = \delta$ and reparametrize as:

$$S_{\mathrm{EE}}^{\mathrm{E}} = \frac{L_\star^{d-1}\pi^{(d-1)/2}}{2G_N\Gamma\left[\frac{d-1}{2}\right]}\int_{\delta/\ell}^1 \mathrm{d}y\, \frac{(1-y^2)^{(d-3)/2}}{y^{d-1}} \ . \tag{5.19}$$

[3]Incidentally, the quadratic character of the anomaly piece has another interesting consequence. If we were to minimize the higher-curvature functional fully non-perturbatively, the RT surface would also be extremal, because the anomaly part contribution to the Euler-Lagrange equation is linear in extrinsic curvatures, and these vanish for the RT surface.

Integrating by parts, it is easy to show that for odd d we get a constant term, while for even d we get a logarithmic one. The final result takes the form

$$S_{\rm EE}^{\rm E} \supset \begin{cases} (-1)^{\frac{d-2}{2}} 4 a_{\rm E}^{\star(d)} \log\left(\frac{\ell}{\delta}\right) & \text{for even } d\,, \\ (-1)^{\frac{d-1}{2}} 2\pi a_{\rm E}^{\star(d)} & \text{for odd } d\,, \end{cases} \tag{5.20}$$

where $\supset$ means we ignore the series of non-universal divergent pieces of the form $(\ell/\delta)^{(d-2k)}$ with $k = 1, 2, \ldots, (d-1)/2$ for odd d and $k = 1, 2, \ldots, (d-2)/2$ for even d – see *e.g.* [32, 33] for the numerical coefficients. The normalization factors are different from those in the general expression (5.5), they are chosen to obtain a single expression for the coefficient:

$$a_{\rm E}^{\star(d)} = \frac{\pi^{\frac{d-2}{2}}}{8\Gamma(d/2)} \frac{L_\star^{d-1}}{G_N}\,. \tag{5.21}$$

As we already mentioned, the vanishing of the extrinsic curvatures makes the result for perturbative higher-curvature theories reduce to the corresponding Wald piece, which in turn reduces to the Einstein gravity result via (5.12). Hence, the universal terms for a general higher-curvature theory look again like (5.20), where $a_{\rm E}^{\star(d)}$ is replaced by a new $a^{\star(d)} = -32\pi G_N k_0 a_{\rm E}^{\star(d)}$. Combining this with (5.11), we get the universal coefficient in terms of the on-shell AdS Lagrangian of the dual bulk theory:

$$a^{\star(d)} = \frac{\pi^{d/2} L_\star^{d+1}}{d\,\Gamma(d/2)} \mathcal{L}_E|_{\rm AdS}\,. \tag{5.22}$$

As argued in the introduction to this chapter, this quantity can be related with the type-A conformal anomaly coefficient in even dimensions, or with the free energy of the CFT in $\mathbb{S}^d$ in odd dimensions.

Let us now present the explicit form of $a^{\star(d)}$ for some families of higher-curvature bulk theories. In this case, the previous relation makes it an easy computation: we do not have to go through the holographic entanglement entropy functional, just evaluating the on-shell Euclidean Lagrangian is enough. For quadratic and cubic theories, following the conventions of the previous chapter (4.99) and (4.105):

$$a_{\rm Riem^2}^{\star(d)} = \left[1 - 2d(d+1)\alpha_1 - 2d\alpha_2 - 4\alpha_3\right] a_{\rm E}^{\star(d)}\,, \tag{5.23a}$$

$$\begin{aligned} a_{\rm Riem^3}^{\star(d)} = \big[&1 + 3(d-1)\beta_1 + 12\beta_2 + 6d\beta_3 + 6d(d+1)\beta_4 + 3d^2\beta_5 + 3d^2\beta_6 \\ &+3d^2(d+1)\beta_7 + 3d^2(d+1)^2\beta_8\big]\, a_{\rm E}^{\star(d)}\,. \end{aligned} \tag{5.23b}$$

For a general Lovelock theory of the form (4.88), one would have:

$$a_{\rm Lovelock}^{\star(d)} = [1 - \sum_n n\,(-1)^n \prod_{k=1}^{2(d-1)} (d-k)\lambda_n] a_{\rm E}^{\star(d)}\,. \tag{5.24}$$

Explicitly, for the quadratic (Gauss-Bonnet) and cubic Lovelock theories:

$$a_{\mathcal{X}_4}^{\star(d)} = [1 - 2(d-2)(d-1)\lambda_2] a_{\rm E}^{\star(d)}\,, \tag{5.25a}$$

$$a_{\mathcal{X}_6}^{\star(d)} = [1 + 3(d-4)(d-3)(d-2)(d-1)\lambda_3] a_{\rm E}^{\star(d)}\,, \tag{5.25b}$$

which vanish below the critical dimension, as they should. Finally, let us consider a couple of extra cubic theories well-studied in the literature. These are Einsteinian Cubic Gravity (ECG) in four bulk dimensions [147–149]:

$$I_E^{\rm ECG} = -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{G}\left[\frac{6}{L^2} + R - \frac{\mu_{\rm ECG}L^4}{8}\mathcal{P}\right] , \tag{5.26}$$

where

$$\mathcal{P} \equiv 12\mathcal{L}_1^{(3)} + \mathcal{L}_2^{(3)} - 12\mathcal{L}_5^{(3)} + 8\mathcal{L}_6^{(3)} ; \tag{5.27}$$

and Quasi-topological Gravity (QTG) in five bulk dimensions [150, 151]:

$$I_E^{\rm QTG} = -\frac{1}{16\pi G_N}\int \mathrm{d}^5x\sqrt{G}\left[\frac{12}{L^2} + R + \frac{7\mu_{\rm QTG}L^4}{4}\mathcal{Z}_5\right] , \tag{5.28}$$

where

$$\mathcal{Z}_5 \equiv \mathcal{L}_1^{(3)} - \frac{9}{7}\mathcal{L}_3^{(3)} + \frac{3}{8}\mathcal{L}_4^{(3)} + \frac{15}{7}\mathcal{L}_5^{(3)} + \frac{18}{7}\mathcal{L}_6^{(3)} - \frac{33}{14}\mathcal{L}_7^{(3)} + \frac{15}{56}\mathcal{L}_8^{(3)} , \tag{5.29}$$

and where we have omitted the Gauss-Bonnet density, which is usually included in the action. These theories define holographic toy models of non-supersymmetric CFTs in $d = 3$ and $d = 4$, respectively. They also have some special properties which make them especially interesting, like the fact that they possess second-order linearized equations on maximally symmetric backgrounds, that they allow for generalizations of the Schwarzschild solution with a single function, *i.e.*, satisfying $G_{tt}G_{rr} = -1$, as well as the fact that the associated thermodynamic properties can be computed fully analytically. The universal terms of the entanglement entropy for a spherical region in these theories are:

$$a^{\star(3)}_{\rm ECG} = [1 + 3\mu_{\rm ECG}]a^{\star(3)}_{\rm E} , \qquad a^{\star(4)}_{\rm QTG} = [1 + 9\mu_{\rm QTG}]a^{\star(4)}_{\rm E} . \tag{5.30}$$

As a final general remark, let us rewrite the result (5.12) as part of the full holographic entanglement entropy functional in an AdS bulk, with k_0 substituted by $a^{\star(d)}$:

$$S(A)|_{\rm AdS} = \frac{1}{4G_N}\frac{a^{\star(d)}}{a^{\star(d)}_{\rm E}}\int_{\Gamma_A}\mathrm{d}^{d-1}y\,\sqrt{h} + S_{\rm Anomaly} . \tag{5.31}$$

This shows that the Wald piece will always be proportional to the Einstein gravity result, with the factor relating them equal to the different combinations of couplings we have been showing for the different explicit theories. Furthermore, for theories without anomaly piece, which based on the results presented in the previous chapter are those corresponding to α_1, α_2, β_5, β_6, β_7, and β_8, the net result is just this overall multiplicative correction to the Einstein gravity entanglement entropy.

5.2 Slab regions

Let us consider now an entangling region consisting of a slab of width ℓ along a particular dimension, $x \in [\ell/2, \ell/2]$, and infinite along the remaining $(d-2)$. Due to the flatness of the entangling surface, in this case the entanglement entropy only has the area-law divergence piece plus a constant universal term:

$$S_{\rm EE} = \xi\frac{L_y^{d-2}}{\delta^{d-2}} - \kappa^{(d)}\frac{L_y^{d-2}}{\ell^{d-2}} , \tag{5.32}$$

where ξ is a non-universal constant, and L_y an IR regulator required to control the divergence produced by the infinite size of the region. As opposed to other universal terms, $\kappa^{(d)}$ does not have any (known) alternative interpretation beyond the entanglement entropy of an infinite slab. For instance, it is not expected to be related to charges characterizing simple local correlators. Previous papers where $\kappa^{(d)}$ was computed for certain holographic higher-curvature gravities include [54], where it was evaluated for quadratic theories in $d = 3$, and [145], where it was computed for Gauss-Bonnet gravity in $d = 4$ fully nonperturbatively using the Jacobson-Myers functional.

We write the AdS$_{d+1}$ metric in coordinates adapted to the translational symmetry of our geometric setup:

$$\mathrm{d}s^2 = \frac{L_\star^2}{z^2}\left[\mathrm{d}\tau^2 + \mathrm{d}z^2 + \mathrm{d}x^2 + \mathrm{d}\vec{y}^2_{d-2}\right] . \tag{5.33}$$

The RT surface can be parametrize as $z = Z(x)$, precisely due to the translational invariance. Unit normals are given by:

$$n_1 = \frac{z}{L_\star}\partial_\tau , \qquad n_2 = \frac{z}{L_\star\sqrt{1+Z'^2}}\left(Z'\partial_x - \partial_z\right) , \tag{5.34}$$

and the induced metric on the surface is:

$$h_{MN}\mathrm{d}x^M\mathrm{d}x^N = \frac{L_\star^2}{Z^2}\left[\frac{1}{1+Z'^2}\left(\mathrm{d}x + Z'\mathrm{d}z\right)^2 + \mathrm{d}\vec{y}^2_{d-2}\right] . \tag{5.35}$$

The non-vanishing components of the extrinsic curvatures read:

$$K^2{}_{xx} = \frac{L_\star\left(1+Z'^2+ZZ''\right)}{Z^2(1+Z'^2)^{5/2}} = \frac{K^2{}_{xz}}{Z'} = \frac{K^2{}_{zz}}{Z'^2} , \quad K^2{}_{mn} = \frac{L_\star\delta_{mn}}{Z^2\sqrt{1+Z'^2}} , \tag{5.36}$$

with $m, n, \ldots$ indices in the $(d-2)$ directions y^m, and all components of $K^1{}_{MN}$ vanish. The Ryu-Takayanagi surface is determined by the condition $K^2 = 0$, where in this case we have

$$K^2 = \frac{(d-1)(1+Z'^2) + ZZ''}{L_\star(1+Z'^2)^{3/2}} . \tag{5.37}$$

A first integral can be shown to exist so that

$$Z' = -\frac{\sqrt{z_\star^{2(d-1)} - Z^{2(d-1)}}}{Z^{(d-1)}} , \quad \text{where} \quad z_\star = \frac{\Gamma\left[\frac{1}{2(d-1)}\right]}{2\sqrt{\pi}\,\Gamma\left[\frac{d}{2(d-1)}\right]}\ell , \tag{5.38}$$

is the value of z corresponding to the turning point of the surface. After some massaging, the EE for Einstein gravity can be seen to be given by [32, 33]:

$$S_{\rm EE}^{\rm E} = \frac{L_\star^{d-1}L_y^{d-2}}{2Gz_\star^{d-2}}\int_\delta^1 \frac{\mathrm{d}y}{y^{d-1}\sqrt{1-y^{2(d-1)}}} = \xi_{\rm E}\frac{L_y^{d-2}}{\delta^{d-2}} - \kappa_{\rm E}^{(d)}\frac{L_y^{d-2}}{\ell^{d-2}} , \tag{5.39}$$

with

$$\kappa_{\rm E}^{(d)} = \frac{2^{d-3}\pi^{\frac{d-1}{2}}\Gamma\left[\frac{d}{2(d-1)}\right]^{d-1}}{(d-2)\Gamma\left[\frac{1}{2(d-1)}\right]^{d-1}}\frac{L_\star^{d-1}}{G_N} , \quad \xi_{\rm E} = \frac{L_\star^{d-1}}{2(d-2)G_N} . \tag{5.40}$$

Let us generalize now this result to quadratic and cubic theories. We only need to obtain the corresponding combinations entering the anomaly from the previously computed extrinsic and bulk curvature tensors. For the general quadratic theory, we have:

$$S_{\rm EE}^{\rm Riem^2}=\frac{L_\star^{d-1}L_y^{d-2}}{2G_N z_\star^{d-2}}\int_\delta^1 \frac{\left[1-2d(d+1)\alpha_1-2d\alpha_2-2\alpha_3\left[2+(d-1)(d-2)y^{2(d-1)}\right]\right]}{y^{d-1}\sqrt{1-y^{2(d-1)}}}\mathrm{d}y\ ,$$

and from this we can integrate to obtain expression (5.32), with universal coefficient given by:

$$\kappa^{(d)}_{\rm Riem^2}=\left[1-2d(d+1)\alpha_1-2d\alpha_2+2(d-3)\left[2+d(d-2)\right]\alpha_3\right]\kappa^{(d)}_{\rm E}\,. \tag{5.41}$$

The non-universal term $\xi_{\rm Riem^2}$ gets a factor with respect to $\xi_{\rm E}$ identical to the one of $a^{\star(d)}_{\rm Riem^2}$ with respect to $a^{\star(d)}_{\rm E}$ of the previous section. Note that there are two kinds of terms in the integrand. On the one hand, pieces arising from purely bulk curvatures are proportional to the Einstein gravity one, which is of the form $\sim 1/(y^{d-1}\sqrt{1-y^{2(d-1)}})$. On the other hand, the contribution which involves two extrinsic curvatures has an extra $\sim y^{2(d-1)}$ factor. It is easy to see that $\xi_{\rm Riem^2}$ is unaffected by the second type of terms, because it comes from contributions close to the boundary $y\to 0$; this explains why it is not sensitive to the anomaly part of the functional. Nevertheless, recall that ξ is not a universal quantity, so its interest is very limited. On the other hand, the universal constant $\kappa^{(d)}_{\rm Riem^2}$ does get affected by the extrinsic curvature term. The result for $\kappa^{(3)}_{\rm Riem^2}$ agrees with the one obtained in [54], as it should.

We find a similar kind of behavior for the cubic theories. Wald-like terms produce contributions proportional to the Einstein gravity result, and the non-universal constant $\xi_{\rm Riem^3}$ satisfies $\xi_{\rm Riem^3}/\xi_{\rm E}=a^{\star(d)}_{\rm Riem^3}/a^{\star(d)}_{\rm E}$. On the other hand, terms with two extrinsic curvatures have an extra factor $\sim y^{2(d-1)}$ in the integrand, and those with four, one of the form $\sim y^{4(d-1)}$. Both types of terms affect the universal coefficient, which takes the form:

$$\begin{aligned}\kappa^{(d)}_{\rm Riem^3}=\Bigg[&1+3\left[d-1+(d-1)(d-2)^2+\frac{d(d-2)^2(8+d(d^2-9))}{8(2d-1)}\right]\beta_1\\&+12\left[1-(d-1)(d-2)^2+\frac{d(d-1)(d-2)^2(7+d(d-5))}{4(2d-1)}\right]\beta_2\\&+2d\Big[3-(d-1)(d-2)^2\Big]\beta_3+2d(d+1)\Big[3-(d-1)(d-2)^2\Big]\beta_4\\&+3d^2\beta_5+3d^2\beta_6+3d^2(d+1)\beta_7+3d^2(d+1)^2\beta_8\Bigg]\kappa^{(d)}_{\rm E}\,.\end{aligned} \tag{5.42}$$

A check of these results can be performed by particularizing them to Lovelock theories. We find:

$$\kappa^{(d)}_{\mathcal{X}_4}=\left[1+2(d-3)(d-2)(d-1)\lambda_2\right]\kappa^{(d)}_{\rm E}\ , \tag{5.43a}$$

$$\kappa^{(d)}_{\mathcal{X}_6}=\left[1-\frac{3(d-5)(d-4)(d-3)(d-2)(d-1)^2}{(2d-1)}\lambda_3\right]\kappa^{(d)}_{\rm E}\ , \tag{5.43b}$$

which vanish in dimensions lower or equal to the critical one, *i.e.*, for $d+1\leq 2n$. One can also verify that $\kappa^{(4)}_{\mathcal{X}_4}$ agrees with the nonperturbative result found in [145] at leading order in λ_2. For ECG and QTG we find, respectively:

$$\kappa^{(3)}_{\rm ECG}=\left[1+3\mu_{\rm ECG}\right]\kappa^{(3)}_{\rm E}\ ,\qquad \kappa^{(4)}_{\rm QTG}=\left[1+9\mu_{\rm QTG}\right]\kappa^{(4)}_{\rm E}\ . \tag{5.44}$$

As mentioned above, the coefficient $\kappa^{(d)}$ does not have an alternative interpretation beyond entanglement entropy, which is manifest in this case from the fact that in all cases in which various coefficients characterizing the dual CFT have been computed for some of the above theories, all the corresponding values differ from the ones obtained here for $\kappa^{(d)}$ – as explained below, the sharp-limit corner coefficient can be shown to coincide with $\kappa^{(3)}$ on general grounds, but this is an exception. This includes, in particular, all the rest of coefficients computed in this chapter (conformal anomaly coefficients in $d = 4$ and $d = 6$; $a^{\star(d)}$ in general d; the corner charge σ in $d = 3$) as well as others like the stress-tensor two-point function charge C_T, the coefficient C_S relating the thermal entropy of a plasma to its temperature, as well as others arising in the context of holographic complexity [54, 116, 122, 145, 146, 151–153].

5.3 Cylinder regions

Let us now consider (hyper)cylindrical entangling surfaces. We will be mostly interested in the universal logarithmic piece arising for such regions in $d = 4$ and $d = 6$ theories, for as we argued at the beginning of the chapter this will allow us to obtain the coefficients of the conformal anomaly. We write the Euclidean $\mathrm{AdS}_{(d+1)}$ metric as:

$$\mathrm{d}s^2 = \frac{L_\star^2}{z^2}\left[\mathrm{d}\tau^2 + \mathrm{d}z^2 + \mathrm{d}\vec{y}^2_{(d-3-J)} + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{(J+1)}\right] \, , \tag{5.45}$$

where $\mathrm{d}\Omega^2_{(J+1)}$ is the metric of a round $(J+1)$-dimensional sphere. Our entangling regions will be given by $\tau = 0$, $r = R_0$ at the boundary $z = 0$. J takes values $J = 0, \ldots, d-3$, which correspond to entangling surfaces $\partial A = \mathbb{S}^1 \times \mathbb{R}^{d-3}, \mathbb{S}^2 \times \mathbb{R}^{d-4}, \ldots, \mathbb{S}^{d-3} \times \mathbb{R}^1, \mathbb{S}^{d-2}$, respectively.

To avoid confusing notation, let us differentiate between the bulk coordinate z and the parameter on the surface, which we will call ζ. We are parametrizing the bulk RT surface as $z = \zeta$ and $r = R(\zeta)$. We also have parameters along the y^m directions, $m = 1, \ldots, d-3-J$; and along the angular directions on the sphere, ϕ^α with $\alpha = 1, \ldots, J+1$. Unit normals to the surface read:

$$n_1 = \frac{z}{L_\star}\partial_\tau \, , \quad n_2 = \frac{z}{L_\star\sqrt{1+R'^2}}\left(R'\partial_z - \partial_r\right) \, . \tag{5.46}$$

The induced metric in the coordinates parameterizing the surface is:

$$h_{ij}\mathrm{d}y^i\mathrm{d}y^j = \frac{L_\star^2}{\xi^2}\left[(1+R'^2)\mathrm{d}\xi^2 + \mathrm{d}\vec{y}^2_{(d-3-J)} + R^2 d\Omega^2_{(j+1)}\right] \, . \tag{5.47}$$

In this same surface coordinates, the non-vanishing components of ${K^2}_{ij}$ read (${K^1}_{ij}$ vanish by time-translation symmetry, just like in previous examples):

$${K^2}_{\zeta\zeta} = \frac{-L_\star(R' + R'^3 - \zeta R'')}{\zeta^2\sqrt{1+R'^2}} \, , \quad {K^2}_{y^m y^n} = \frac{-L_\star R'\delta_{mn}}{\zeta^2\sqrt{1+R'^2}} \, , \tag{5.48a}$$

$${K^2}_{\phi^\alpha\phi^\beta} = \frac{-L_\star(\zeta + RR')R}{\zeta^2\sqrt{1+R'^2}}\prod_{\gamma=1}^{J-1}\sin^2\phi^\gamma\delta_{\alpha\beta} \, . \tag{5.48b}$$

The equation for the RT surface is, as usual, $K^2 = 0$, where

$$K^2 = \frac{(RR'' - (J+1))\zeta - (d-1)RR'(1+R'^2) - (J+1)\zeta R'^2}{L_\star R(1+R'^2)^{3/2}} \ . \tag{5.49}$$

In the case of Einstein gravity, the RT functional reduces to:

$$S_{\rm EE}^{\rm E} = \frac{L_\star^{d-1} L_y^{d-3-J}\Omega_{(J+1)}}{4G_N}\int_\delta^{z_{\rm max}} {\rm d}\zeta \frac{R^{J+1}}{\zeta^{d-1}}\sqrt{1+R'^2} \ , \tag{5.50}$$

where $\Omega_{(J+1)} \equiv 2\pi^{(J+2)/2}/\Gamma[(J+2)/2]$. If we are only interested in the logarithmically divergent contribution to the entanglement entropy in even dimensional theories, we can obtain it via a near-boundary analysis, because such contribution is local in the entangling surface ∂A. This means it suffices to consider a perturbative solution to $K^2 = 0$ near $\zeta = 0$. The result reads:[4]

$$R(\zeta) = R_0 - \frac{(J+1)}{2R_0(d-2)}\zeta^2 + \mathcal{O}(\zeta^4) \ , \tag{5.51}$$

which is the function we need to plug back into our functionals. Let us separately study the cases of four and six-dimensional boundaries.

5.3.1 Four dimensions

For general CFTs in four dimensions, the universal coefficient of the logarithmic divergence in the entanglement entropy of a region with a smooth entangling surface is given by Solodukhin's formula, (5.9). As argued there, the a-coefficient of the conformal anomaly is obtained by considering $\partial A = \mathbb{S}^2$, which we already did when discussing spherical entangling regions. On the contrary, the c-coefficient can be obtained via a cylindrical region $\partial A = \mathbb{R}\times\mathbb{S}^1$. We will obtain c now, for which we take $J = 0$ and $d = 4$ in the previous expressions.

If we consider the Einstein gravity functional, (5.50), we can isolate the universal term via a near-boundary analysis as:

$$S_{\rm EE}^{\rm E} = \frac{\pi L_\star^3 L_y}{2G_N}\int_\delta^{z_{\rm max}} {\rm d}\zeta \left[\frac{R_0}{\zeta^3} - \frac{1}{8R_0\zeta} + \ldots\right] = \ldots - \frac{c_{\rm E}}{2}\frac{L_y}{R_0}\log\left(R_0/\delta\right) + \ldots \ , \tag{5.52}$$

where

$$c_{\rm E} = \frac{\pi L_\star^3}{8G_N} \ . \tag{5.53}$$

This takes the form expected for a cylinder region in general CFTs, where the value of $c_{\rm E}$ matches the corresponding trace anomaly charge. In our conventions, this is in turn related to the stress-tensor two-point function charge C_T through $c = \pi^4 C_T/40$ for general theories. Computing $C_T^{\rm E}$ by a different method – see (5.56) for the value and the discussion about how it can be obtained –, we see that it perfectly matches with the previous value for $c_{\rm E}$.

[4]When performing this expansion, it does not seem to be possible to solve the equation beyond quadratic order for $d = 4$, and beyond quartic order in $d = 6$. While this does not affect our calculations, it would be interesting to better understand the origin of this issue.

Performing the analogous calculations for quadratic and cubic theories, we observe that introducing the expansion (5.51) in the corresponding functionals there are three kinds of terms which appear multiplying the Einstein gravity integrand in (5.52): terms coming from the Wald piece, which are constant; terms involving products of two extrinsic curvatures, which are $\sim \zeta^2$; and terms involving products of four extrinsic curvatures, which go with $\sim \zeta^4$. Terms of the latter kind do not contribute to c, which is a manifestation of the splitting-independent nature of this coefficient. The final result for c_{Riem^2} and c_{Riem^3} reads:

$$c_{\mathrm{Riem}^2} = [1 - 40\alpha_1 - 8\alpha_2 + 4\alpha_3]\, c_{\mathrm{E}} \,, \tag{5.54a}$$

$$c_{\mathrm{Riem}^3} = [1 + 21\beta_1 - 36\beta_2 - 8\beta_3 - 40\beta_4 + 48\beta_5 + 48\beta_6 + 240\beta_7 + 1200\beta_8]\, c_{\mathrm{E}} \,. \tag{5.54b}$$

These are again in agreement with the general relation with C_T. Indeed, for general quadratic and cubic theories in d-dimensions one finds [124]:

$$C_T^{\mathrm{Riem}^2} = [1 - 2d(d+1)\alpha_1 - 2d\alpha_2 + 4(d-3)\alpha_3]\, C_T^{\mathrm{E}} \,, \tag{5.55a}$$

$$\begin{aligned} C_T^{\mathrm{Riem}^3} = &[1 + 3(3d-5)\beta_1 - 12(2d-5)\beta_2 - 2d(2d-7)\beta_3 - 2d(2d-7)(d+1)\beta_4 \\ &+3d^2\beta_5 + 3d^2\beta_6 + 3d^2(d+1)\beta_7 + 3d^2(d+1)^2\beta_8]\, C_T^{\mathrm{E}} \,, \end{aligned} \tag{5.55b}$$

where the Einstein gravity result reads:

$$C_T^{\mathrm{E}} = \frac{\Gamma[d+2]}{8(d-1)\Gamma[d/2]\pi^{\frac{d+2}{2}}} \frac{L_\star^{d-1}}{G_N} \,. \tag{5.56}$$

These results for C_T can be obtained in different ways. A simple one consists in computing the linearized equations of the theory around an AdS background. For a general higher-curvature gravity, these are fourth-order equations which describe the dynamics of a massive scalar mode and a ghost-like massive graviton, in addition to the usual general relativity massless graviton. The resulting equations can be characterized in terms of the masses of the new two modes as well as an effective Newton constant. This generically takes the form $G_{\mathrm{eff}} = G_N/\gamma$, where γ depends on the higher-curvature couplings. Via holography, a rescaling of G_N is equivalent to a rescaling of the stress-tensor charge C_T, which becomes γC_T^{E}. G_{eff} was computed in [124] explicitly for general quadratic, cubic and quartic gravities in general dimensions, so we can easily obtain the values of C_T shown above. In the particular cases of Lovelock, Quasi-topological and Einsteinian cubic gravity densities, the previous values reduce to:

$$C_T^{\mathcal{X}_4} = [1 - 2(d-2)(d-3)\lambda_2]\, C_T^{\mathrm{E}} \,, \tag{5.57a}$$

$$C_T^{\mathcal{X}_6} = [1 + 3(d-2)(d-3)(d-4)(d-5)\lambda_3]\, C_T^{\mathrm{E}} \,, \tag{5.57b}$$

$$C_T^{\mathrm{QTG}} = [1 - 3\mu_{\mathrm{QTG}}]\, C_T^{\mathrm{E}} \,, \tag{5.57c}$$

$$C_T^{\mathrm{ECG}} = [1 - 3\mu_{\mathrm{ECG}}]\, C_T^{\mathrm{E}} \,. \tag{5.57d}$$

Note that all these differ from the slab coefficients $\kappa^{(d)}$ computed in the previous section.

5.3.2 Six dimensions

Let us now turn to six dimensions. In this case, a similar expression to (5.9) for the logarithmic term involving the trace anomaly coefficients holds for general CFTs, and is given by [116, 146]:

$$c_0^{6d} = \int_{\partial A} \left[2A\mathcal{X}_4 + \frac{3\pi}{2} B_1(3T_1 - 2T_2) - 12\pi B_2 T_2 + 6\pi B_3(T_3 + 9T_1 - 12T_2) \right] , \tag{5.58}$$

where $\mathcal{X}_4$ is the Euler density associated to the induced metric on ∂A and now

$$T_1 \equiv (\mathrm{Tr}\, k^2)^2 - \frac{1}{2}k^2 \mathrm{Tr}\, k^2 + \frac{1}{16}k^4 , \tag{5.59a}$$

$$T_2 \equiv \mathrm{Tr}\, k^4 - k\mathrm{Tr}\, k^3 + \frac{3}{8}k^2 \mathrm{Tr}\, k^2 - \frac{3}{64}k^4 , \tag{5.59b}$$

$$T_3 \equiv (\nabla_\alpha k)^2 - \frac{25}{16}k^4 + 11k^2 \mathrm{Tr}\, k^2 - 6(\mathrm{Tr}\, k^2)^2 - 16k\mathrm{Tr}\, k^3 + 12\mathrm{Tr}\, k^4 . \tag{5.59c}$$

In these expressions, we are assuming the region A to be in a fixed time slice of flat space, so that there is only one non-vanishing extrinsic curvature to the surface ∂A. Then, $\mathrm{Tr}\, k^n = k_{\mu_1}^{\mu_2} k_{\mu_2}^{\mu_3} \dots k_{\mu_n}^{\mu_1}$, $k = k_\mu^\mu$, and ∇_α is the derivative associated with the induced metric in ∂A – we take α an index in ∂A. The coefficients A, B_1, B_2 and B_3 are the ones appearing in the trace anomaly, which in this case takes the form:

$$\langle T_\mu^\mu \rangle = \sum_{i=1}^{3} B_i I_i + 2A\mathcal{X}_6 , \tag{5.60}$$

where $\mathcal{X}_6$ is the Euler density, and the I_i are cubic conformal invariants given by:

$$I_1 \equiv C_{\sigma\mu\nu\rho} C^{\mu\lambda\tau\nu} C_\lambda{}^{\sigma\rho}{}_\tau , \tag{5.61a}$$

$$I_2 \equiv C_{\mu\nu}{}^{\rho\sigma} C_{\rho\sigma}{}^{\lambda\tau} C_{\lambda\tau}{}^{\mu\nu} , \tag{5.61b}$$

$$I_3 \equiv C_{\mu\rho\sigma\lambda} \left(\nabla^2 \delta_\nu^\mu + 4R_\nu^\mu - \frac{6}{5} R \delta_\nu^\mu \right) C^{\nu\rho\sigma\lambda} . \tag{5.61c}$$

For the cylindrical entangling regions we are considering here, the induced metric on $d = 6$ Minkowski space reads:

$$\mathrm{d}s^2_{\partial A} = \mathrm{d}\vec{y}^2_{(3-J)} + R_0^2 \mathrm{d}\Omega^2_{(J+1)} . \tag{5.62}$$

The relevant expressions for the extrinsic curvature invariants are:

$$k = \frac{(J+1)}{R_0} , \quad \mathrm{Tr}\, k^n = \frac{(J+1)}{R_0^n} , \tag{5.63}$$

and from this, one finds

$$\mathcal{X}_4 = \frac{(J-2)(J-1)J(J+1)}{R_0^4} , \tag{5.64a}$$

$$T_1 = \frac{(J-3)^2(J+1)^2}{16R_0^4} , \tag{5.64b}$$

$$T_2 = -\frac{(J-3)(J+1)(7+3J(J-2))}{64R_0^4} , \tag{5.64c}$$

$$T_3 = \frac{(J-3)(J+1)(3+J(26-25J))}{16R_0^4} , \tag{5.64d}$$

where, for completeness, we also included the value of $\mathcal{X}_4$, which vanishes for all the cylinder-like regions ($J = 0, 1, 2$). Then, the entanglement entropy universal term reduces, for general CFTs, to

$$c_0^{6d} = \frac{(J+1)\Omega_{(J+1)}}{64}\Big\{128AJ(J-1)(J-2) + 3\pi(J-3)\big[B_1(9J(J-2)-11) \\ + 4B_2(3J(J-2)+7) - 8B_3(J+1)(3+7J)\big]\Big\}\frac{L_y^{3-J}}{R_0^{3-J}} \, . \tag{5.65}$$

Let us now turn to the holographic calculation. The result for Einstein gravity reads, using (5.51) in (5.50):

$$\begin{aligned} S_{\text{EE}}^{\text{E}} &= \frac{L_\star^5 L_y^{3-J}\Omega_{(J+1)}}{4G_N}\int_\delta^{z_{\max}} \mathrm{d}\zeta \left[\frac{R_0^{J+1}}{\zeta^5} - \frac{3(J+1)^2 R_0^{J-1}}{32\zeta^3} + \frac{(J+1)^3(7J-9)}{2048 R_0^{3-J}\zeta} + \ldots\right] \\ &= \ldots + \frac{(J+1)^3(7J-9)\Omega_{(J+1)}L_\star^5}{8192 G_N}\frac{L_y^{3-J}}{R_0^{3-J}}\log\left(\frac{\ell}{\delta}\right) + \ldots \, . \end{aligned} \tag{5.66}$$

Comparing with (5.65) for $J = 0, 1, 2, 3$, we can obtain the Einstein gravity values of A, B_1, B_2, B_3. The results read:

$$A_{\text{E}} = \frac{L_\star^5}{512 G_N} \, , \quad B_1^{\text{E}} = -\frac{L_\star^5}{256\pi G_N} \, , \quad B_2^{\text{E}} = -\frac{L_\star^5}{1024\pi G_N} \, , \quad B_3^{\text{E}} = \frac{L_\star^5}{3072\pi G_N} \, , \tag{5.67}$$

in agreement with previous calculations [146, 154]. In particular, the value of the A charge satisfies $A_{\text{E}} = a_{\text{E}}^{\star(6)}/(32\pi^2)$, a relation which holds for general theories in the present conventions. In particular, the values of A for all the rest of holographic higher-curvature theories are proportional to the corresponding coefficients $a^{\star(6)}$.

Moving to quadratic theories, the contributions without anomaly piece modify the charges in the same way as $a^{\star(6)}$, whereas the term involving two Riemanns contains an extra piece coming from a contraction of extrinsic curvatures, which in this case reads:

$$K_{aij}K^{aij} = -\frac{(J-3)(J+1)}{4L_\star^2 R_0^2}\zeta^2 + \frac{(J-3)^2(J+1)^2}{64 L_\star^2 R_0^4}\zeta^4 + \ldots \, . \tag{5.68}$$

Putting the pieces together in the quadratic functional and comparing with (5.65), we find:

$$B_1^{\text{Riem}^2} = \left[1 - 84\alpha_1 - 12\alpha_2 + \frac{4}{3}\alpha_3\right] B_1^{\text{E}} \, , \tag{5.69a}$$

$$B_2^{\text{Riem}^2} = \left[1 - 84\alpha_1 - 12\alpha_2 - \frac{28}{3}\alpha_3\right] B_2^{\text{E}} \, , \tag{5.69b}$$

$$B_3^{\text{Riem}^2} = \left[1 - 84\alpha_1 - 12\alpha_2 + 12\alpha_3\right] B_3^{\text{E}} \, . \tag{5.69c}$$

It can be easily verified that these results reduce to the ones found in [155] for seven-dimensional Critical Gravity [156, 157]. It is also easy to check that the previous charges satisfy the relation $3B_3 = (B_2 - B_1/2)$, which holds for theories that are unaffected by the splittings choice, as argued in [116].

Proceeding analogously with the cubic densities, we obtain:

$$
\begin{aligned}
B_1^{\text{Riem}^3} &= [1 + 39\beta_1 - 20\beta_2 + 4\beta_3 + 28\beta_4 \\
&\qquad +108\beta_5 + 108\beta_6 + 756\beta_7 + 5292\beta_8]\, B_1^{\text{E}} \,, \qquad (5.70\text{a}) \\
B_2^{\text{Riem}^3} &= [1 + 7\beta_1 - 20\beta_2 + 68\beta_3 + 476\beta_4 \\
&\qquad +108\beta_5 + 108\beta_6 + 756\beta_7 + 5292\beta_8]\, B_2^{\text{E}} \,, \qquad (5.70\text{b}) \\
B_3^{\text{Riem}^3} &= [1 + 39\beta_1 - 84\beta_2 - 60\beta_3 - 420\beta_4 \\
&\qquad +108\beta_5 + 108\beta_6 + 756\beta_7 + 5292\beta_8]\, B_3^{\text{E}} \,. \qquad (5.70\text{c})
\end{aligned}
$$

We can check, at this order, which theories satisfy the $3B_3 - (B_2 - B_1/2) = 0$ condition. Evaluating the quantity in the left-hand side, one obtains

$$
3B_3 - (B_2 - B_1/2) = -\frac{(\beta_1 + 2\beta_2)L_\star^2}{32\pi G_N} \,. \qquad (5.71)
$$

Hence, such a combination vanishes for all theories for which $\beta_1 = -2\beta_2$. This includes, in particular, the cubic Lovelock density, in agreement with the result of [146]. The explicit expressions for the quadratic and cubic Lovelock theories read:

$$
B_1^{\mathcal{X}_4} = \left[1 - \frac{104}{3}\lambda_2\right] B_1^{\text{E}} \,, \quad B_2^{\mathcal{X}_4} = \left[1 - \frac{136}{3}\lambda_2\right] B_2^{\text{E}} \,, \quad B_3^{\mathcal{X}_4} = [1 - 24\lambda_2]\, B_3^{\text{E}} \,, \quad (5.72\text{a})
$$

$$
B_1^{\mathcal{X}_6} = [1 + 136\lambda_3]\, B_1^{\text{E}} \,, \qquad B_2^{\mathcal{X}_6} = [1 + 200\lambda_3]\, B_2^{\text{E}} \,, \qquad B_3^{\mathcal{X}_6} = [1 + 72\lambda_3]\, B_3^{\text{E}} \,. \quad (5.72\text{b})
$$

5.4 Corner regions

5.4.1 General aspects of corner entanglement

The structure of divergences and universal terms in the entanglement entropy gets modified when the entangling surface ∂A contains geometric singularities – see *e.g.*, [158, 159] for some general accounts of this phenomenon in various dimensions. Here, we will focus on the archetypical example of (straight) corners in $d = 3$ CFTs. Given a fixed time slice, the entanglement entropy corresponding to a corner region of opening angle θ in the ground state of a CFT regulated by a short distance cutoff δ takes the form:

$$
S_{\text{EE}} = b_1 \frac{H}{\delta} - a(\theta) \log(H/\delta) + b_0 \,, \qquad (5.73)
$$

where H is an IR regulator, $a(\theta)$ is known as the corner function, b_1 is a non-universal coefficient, and b_0 is a coefficient which generically contains a universal non-local contribution and a non-universal part of intrinsically local nature induced by possible redefinitions of the regulator δ. With respect to the case of smooth regions, the novelty here is the appearance of a new logarithmic divergence controlled by the corner function, of universal nature. Many aspects of this function have been studied in a plethora of contexts – to cite some of the extensive literature on the subject, see [54, 160–165]. As a result of this work, the function $a(\theta)$ has been shown to satisfy a number of properties, universal relations and bounds which we summarize now.

On the one hand, the purity of the ground state, which implies the well-known relation $S_{\rm EE}(A) = S_{\rm EE}(\bar{A})$, requires $a(\theta) = a(2\pi - \theta)$. Besides, using strong subadditivity and Lorentz invariance one can show that [161]:

$$a(\theta) \geq 0 \ , \quad \partial_\theta a(\theta) \leq 0 \ , \quad \partial_\theta^2 a(\theta) \geq -\frac{\partial_\theta a(\theta)}{\sin\theta} \ , \quad \text{for} \quad \theta \in [0, \pi] \ . \tag{5.74}$$

In particular, this implies that $a(\theta)$ is a positive, monotonously-decreasing and convex function of the opening angle as we vary it from $\theta \sim 0$, corresponding to a very sharp corner, to $\theta \sim \pi$, corresponding to a very open, almost-smooth one. In those two limits, the function behaves, respectively, as [160–162]:

$$a(\theta \simeq 0) = \frac{\kappa}{\theta} + \mathcal{O}(\theta) \ , \qquad a(\theta \simeq \pi) = \sigma \cdot (\theta - \pi)^2 + \sum_{p=2} \sigma^{(p-1)} \cdot (\theta - \pi)^{2p} \ . \tag{5.75}$$

In the first expression, κ is a constant which can be shown to coincide with the slab coefficient $\kappa^{(3)}$ – see (5.32) above – for general theories [54, 158]. In the second formula, we have made manifest the fact that only even powers appear in the expansion. The leading coefficient, σ, turns out to be related to the stress-energy tensor two-point function coefficient C_T through

$$\sigma = \frac{\pi^2}{24} C_T \ , \tag{5.76}$$

for general CFTs [166]. In fact, the full corner functions of all CFTs considered so far in the literature turn out to become very close to each other when normalized by C_T [164]. Using (5.76) and the third relation in (5.74), a lower bound on $a(\theta)$ valid for general CFTs was obtained in [167]. This takes the form:

$$a(\theta) \geq \mathfrak{a}_{\rm min}(\theta) \ , \quad \text{where} \quad \mathfrak{a}_{\rm min}(\theta) \equiv \frac{\pi^2 C_T}{3} \log\left[1/\sin(\theta/2)\right] \ , \tag{5.77}$$

where C_T is to be understood as the one corresponding to the theory we are comparing with. The bound turns out to be pretty tight for all theories considered so far.

The results mentioned so far are valid for general CFTs. Theories for which $a(\theta)$ has been actually computed for general values of the opening angle are nonetheless scarce. For free scalars and fermions, $a(\theta)$ was obtained numerically from a complicated set of coupled differential and algebraic equations in [160–162]. In addition, the Ryu-Takayanagi prescription allowed for the computation of the corresponding corner function for holographic theories dual to Einstein gravity; we will briefly review this result below. The only two cases for which a completely explicit expression for $a(\theta)$ is known correspond, respectively, to certain Lifshitz quantum critical points [168] and the so-called "Extensive Mutual Information model" [169–171]. The corresponding corner functions read:

$$a_{\rm Lif.}(\theta) = \frac{(\theta - \pi)^2}{\theta(2\pi - \theta)} \ , \qquad a_{\rm EMI}(\theta) = 1 + (\pi - \theta)\cot\theta \ . \tag{5.78}$$

Using these two functions, it is possible to construct a simple approximation to the corner function of any CFT provided one knows the values of the corresponding sharp and smooth coefficients, κ and σ. This is given by [163]:

$$\tilde{a}(\theta) = \frac{2\pi(\kappa - 3\pi\sigma)}{\pi^2 - 6} \frac{(\theta - \pi)^2}{\theta(2\pi - \theta)} - \frac{3(2\kappa - \pi^3\sigma)}{\pi(\pi^2 - 6)} \left[1 + (\pi - \theta)\cot\theta\right] \ . \tag{5.79}$$

This respects the asymptotic behavior both as $\theta \to 0$ and as $\theta \to \pi$, and produces very precise approximations to the actual free-field and Einstein gravity results. In all cases, the relative agreement is always better than 99% for all values of θ.

5.4.2 Holographic computation of corner functions

Let us obtain now $a(\theta)$ by means of holographic methods. We start by reviewing how it is done for Einstein gravity: since higher-curvature contributions will be treated perturbatively, the RT surface is an essential input also for those more general cases. First, it is useful to write the AdS_4 metric as:

$$ds^2 = \frac{L_\star^2}{z^2}[\mathrm{d}\tau^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2] \,. \tag{5.80}$$

The corner region is defined by $\tau = 0$, $r \geq 0$, $|\phi| \leq \theta/2$. We can parametrize the bulk surface with two parameters (ρ, ψ) as:

$$r(\rho,\psi) = \rho \,, \qquad \phi(\rho,\psi) = \psi \,, \qquad z(\rho,\psi) = \rho h(\psi) \,, \tag{5.81}$$

where $h(\psi)$ is a function satisfying $h(\psi \to \pm\theta/2) \to 0$. Unit normals to the surface are given by:

$$n_1 = \frac{z}{L_\star}\partial_\tau \,, \quad n_2 = \frac{z}{L_\star\sqrt{1+h^2+\dot{h}^2}}\left[\partial_z - h\partial_r - \frac{\dot{h}}{r}\partial_\phi\right] \,, \tag{5.82}$$

where we have continuously extended to a neiborhood of the surface, h and its derivative $\dot{h}$ being evaluted at a general angle ϕ. Using these we have:

$$\begin{aligned} h_{MN}\mathrm{d}x^M\mathrm{d}x^N = \frac{L_\star^2}{z^2}\Bigg[\mathrm{d}\tau^2 + &\frac{1}{(1+h^2+\dot{h}^2)}\Big[(h^2+\dot{h}^2)\mathrm{d}z^2 + (1+\dot{h}^2)\mathrm{d}r^2 \\ &+ r^2(1+h^2)\mathrm{d}\phi^2 + 2h\,\mathrm{d}z\,\mathrm{d}r + 2r\dot{h}\,\mathrm{d}z\,\mathrm{d}\phi - 2rh\dot{h}\,\mathrm{d}r\,\mathrm{d}\phi\Big]\Bigg] \,. \end{aligned} \tag{5.83}$$

In terms of intrinsic surface coordinates:

$$h_{ij}\mathrm{d}y^i\mathrm{d}y^j = \frac{L_\star^2}{\rho^2h^2}\left[(1+h^2)\mathrm{d}\rho^2 + \rho^2(1+\dot{h}^2)\mathrm{d}\psi^2 + 2\rho h\dot{h}\,\mathrm{d}\rho\,\mathrm{d}\psi\right] \,. \tag{5.84}$$

The non-vanishing components of the extrinsic curvatures are in turn given by:

$$K^2{}_{\rho\rho} = \frac{-L_\star(1+h^2)}{\rho^2h^2\sqrt{1+h^2+\dot{h}^2}} \,, \quad K^2{}_{\rho\psi} = \frac{-L_\star\dot{h}}{\rho h\sqrt{1+h^2+\dot{h}^2}} \,, \tag{5.85}$$

$$K^2{}_{\psi\psi} = \frac{-L_\star(1+h^2+\dot{h}^2+\ddot{h}h)}{h^2\sqrt{1+h^2+\dot{h}^2}} \,. \tag{5.86}$$

These are all the pieces we will need to evaluate the corner function for perturbative higher-order gravities.

For our parametrization of the holographic entangling surface, the Ryu-Takayanagi functional becomes:

$$S_{\mathrm{EE}}^{\mathrm{E}} = \frac{L_\star^2}{2G_N}\int_{\delta/h_0}^{H}\frac{\mathrm{d}\rho}{\rho}\int_0^{\theta/2-\epsilon}\mathrm{d}\psi\frac{\sqrt{1+h^2+\dot{h}^2}}{h^2} \,, \tag{5.87}$$

where we already made manifest the UV cutoff at $z = \delta$, and where $h_0 \equiv h(0)$ is the maximum value taken by the function $h(\psi)$. Also, the angular cutoff ϵ is defined through the condition $\rho h(\theta/2 - \epsilon) = \delta$, which means that the integral over ρ cannot be performed without doing the angular one first, because ϵ is ρ-dependent. The extremal surface condition, $K^2 = 0$, reads:

$$2 + 3h^2 + h^4 + 2\dot{h}^2 + h(1 + h^2)\ddot{h} = 0 \ . \tag{5.88}$$

This has a first integral,

$$\frac{1 + h^2}{h^2\sqrt{1 + h^2 + \dot{h}^2}} = \frac{\sqrt{1 + h_0^2}}{h_0^2} \ , \tag{5.89}$$

which can be used to write $\dot{h}$ in terms of h in the RT functional. Trading the integral over ψ by one over h, and making the change of variables $y = \sqrt{1/h^2 - 1/h_0^2}$, we are left with:

$$\begin{aligned} S_{\mathrm{EE}}^{\mathrm{E}} &= \frac{L_\star^2}{2G_N}\int_{\delta/h_0}^{H}\frac{\mathrm{d}\rho}{\rho}\int_0^{\sqrt{(\rho/\delta)^2 - 1/h_0^2}}\mathrm{d}y\sqrt{\frac{1 + h_0^2(1 + y^2)}{2 + h_0^2(1 + y^2)}} \\ &= \frac{L_\star^2}{2G_N}\int_{\delta/h_0}^{H}\frac{\mathrm{d}\rho}{\rho}\int_0^{\infty}\mathrm{d}y\left[\sqrt{\frac{1 + h_0^2(1 + y^2)}{2 + h_0^2(1 + y^2)}} - 1\right] + \frac{L_\star^2}{2G_N}\int_{\delta/h_0}^{H}\frac{\mathrm{d}\rho}{\rho}\sqrt{\frac{\rho^2}{\delta^2} - \frac{1}{h_0^2}} \ . \end{aligned} \tag{5.90}$$

Expanding this expression for small δ one finally obtains:

$$S_{\mathrm{EE}}^{\mathrm{E}} = \frac{L_\star^2}{2G_N}\frac{H}{\delta} - a_{\mathrm{E}}(\theta)\log(H/\delta) + \mathcal{O}(\delta^0) \ , \tag{5.91}$$

in agreement with the general expression (5.73). The result for the Einstein gravity corner function can be written as:

$$a_{\mathrm{E}}(\theta) = \frac{L_\star^2}{2G_N}\int_0^{+\infty}\mathrm{d}y\left[1 - \sqrt{\frac{1 + h_0^2(1 + y^2)}{2 + h_0^2(1 + y^2)}}\right] \ , \tag{5.92a}$$

$$\theta = \int_0^{h_0}\mathrm{d}h\frac{2\sqrt{1 + h_0^2}\,h^2}{\sqrt{1 + h^2}\sqrt{(h_0^2 - h^2)(h_0^2 + (1 + h_0^2)h^2)}} \ , \tag{5.92b}$$

where the dependence on the opening angle follows implicitly from the relation $h_0(\theta)$ determined by the second integral. It can be verified that $a_{\mathrm{E}}(\theta)$ satisfies all properties explained in the previous subsection. Values of the opening angle close to $\theta = \pi$ correspond to $h_0 \to \infty$, and an expansion of the $\theta(h_0)$ integral in that case can be obtained and inverted, giving:

$$\begin{aligned} h_0 = &\left(\frac{\pi}{\pi - \theta}\right) - \frac{3}{4}\left(\frac{\pi - \theta}{\pi}\right) - \frac{11}{64}\left(\frac{\pi - \theta}{\pi}\right)^3 \\ &- \frac{17}{256}\left(\frac{\pi - \theta}{\pi}\right)^5 - \frac{383}{16384}\left(\frac{\pi - \theta}{\pi}\right)^7 + \mathcal{O}(\pi - \theta)^9 \ . \end{aligned} \tag{5.93}$$

Inserting this in $a_{\mathrm{E}}(\theta)$, one obtains an expansion of the form of the second expression in (5.75), where the leading smooth-limit coefficients are given by:

$$\sigma_{\mathrm{E}} = \frac{L_\star^2}{8\pi G_N}\,, \qquad \sigma'_{\mathrm{E}} = \frac{5L_\star^2}{64\pi^3 G_N}\,, \qquad \sigma''_{\mathrm{E}} = \frac{37L_\star^2}{512\pi^5 G_N}\,, \tag{5.94a}$$

$$\sigma'''_{\mathrm{E}} = \frac{585L_\star^2}{8192\pi^7 G_N}\,, \qquad \sigma^{(4)}_{\mathrm{E}} = \frac{9399L_\star^2}{131072\pi^9 G_N}\,, \qquad \ldots . \tag{5.94b}$$

On the other hand, the sharp limit coefficient is given by [54]:

$$\kappa_{\mathrm{E}} = \frac{L_\star^2}{2\pi G_N}\Gamma[3/4]^4\,. \tag{5.95}$$

Quadratic theories

As observed in [54], the only modification produced on the Einstein gravity corner function $a_{\mathrm{E}}(\theta)$ which arises from including quadratic or $f(R)$ terms in the gravitational action is an overall constant shift. In particular, for the general quadratic theory, one finds:

$$a_{\mathrm{Riem}^2}(\theta) = [1 - 24\alpha_1 - 6\alpha_2]\; a_{\mathrm{E}}(\theta)\,. \tag{5.96}$$

Hence, no new functional dependence on the opening angle is found from these gravitational interactions. The reason for this can be easily understood. Terms in the holographic entanglement entropy functional involving bulk curvatures will reduce to a constant times the Einstein gravity vale, so these clearly cannot produce a new functional dependence on the angle θ. The piece $K_{aij}K^{aij}$, coming from the anomaly part of the $R_{MNRS}R^{MNRS}$ Lagrangian, could in principle generate something new, but it can also be deduced not to contribute from the fact that we can replace $R_{MNRS}R^{MNRS}$ by the Gauss-Bonnet density (plus additional R^2 and $R_{MN}R^{MN}$ terms). The contribution to the holographic entanglement entropy functional of the Gauss-Bonnet invariant is the intrinsic Ricci scalar on the RT surface, which is a topological term in $(d-1) = 2$ dimensions and therefore makes no contribution to the equations of motion. In this case, it does not even modify the Einstein gravity result by an overall constant.

Cubic theories

Let us then consider cubic theories, to see if we can obtain new corner functions. If we only turn on the couplings corresponding to $\mathcal{L}_i^{(3)}$ with $i = 3,4,5,6,7,8$ we find that, similarly to the quadratic case, the corner function is the same as for Einstein gravity up to an overall factor. In the $i = 5,6,7,8$ cases, the fact that the functionals have no anomaly contribution implies that the overall coefficient correcting the Einstein gravity result is the same as for $a^{\star(3)}$, (5.25). For $i = 3,4$, even though there is no modification in the functional dependence of the corner function, there is a modification to the overall coefficient coming from the anomaly terms. The result for all these densities reads:

$$a_{\mathcal{L}^{(3)}_{(3,4,5,6,7,8)}}(\theta) = [1 + 6\beta_3 + 24\beta_4 + 27\beta_5 + 27\beta_6 + 108\beta_7 + 432\beta_8]\, a_{\mathrm{E}}(\theta)\,. \tag{5.97}$$

On the other hand, $\mathcal{L}_1^{(3)}$ and $\mathcal{L}_2^{(3)}$ do modify the angular dependence of $a_{\rm E}$. Keeping only those two terms in the action, we find:

$$a_{\mathcal{L}^{(3)}_{(1,2)}}(\theta) = [1 + 6\beta_1 + 12\beta_2] a_{\rm E}(\theta) + \sum_{i=1}^{2} \beta_i g_i(\theta) \, , \tag{5.98}$$

where

$$g_1(\theta) \equiv + \frac{L_\star^2}{2G_N} \int_0^{+\infty} \frac{3(1+h_0^2)\left[3 + h_0^2(5+4y^2) + 2h_0^4(1+y^2)^2\right]}{\left[1+h_0^2(1+y^2)\right]^{7/2} \sqrt{2+h_0^2(1+y^2)}} \mathrm{d}y \ , \tag{5.99a}$$

$$g_2(\theta) \equiv - \frac{L_\star^2}{2G_N} \int_0^{+\infty} \frac{6(1+h_0^2)\left[3 + h_0^2(7+8y^2) + 4h_0^4(1+y^2)^2\right]}{\left[1+h_0^2(1+y^2)\right]^{7/2} \sqrt{2+h_0^2(1+y^2)}} \mathrm{d}y \ . \tag{5.99b}$$

Hence, at cubic order we find the first examples of holographic corner functions which modify the angular dependence of $a(\theta)$ in a nontrivial way with respect to the Einstein gravity case.

As we mentioned earlier, the almost-smooth limit of the corner function is controlled by C_T for all CFTs. For cubic theories, the result for this coefficient appears in (5.55) above. In $d = 3$, one finds:

$$C_T^{\rm Riem^3} = [1 + 12\beta_1 - 12\beta_2 + 6\beta_3 + 24\beta_4 + 27\beta_5 + 27\beta_6 + 108\beta_7 + 432\beta_8] \, C_T^{\rm E} \ , \tag{5.100}$$

where $C_T^{\rm E} = 3L^2/(\pi^3 G_N)$. Now, including all cubic terms in the action, we find for the smooth limit of $a_{\rm Riem^3}(\theta)$ that indeed

$$\sigma_{\rm Riem^3} = \frac{\pi^2}{24} C_T^{\rm Riem^3} , \tag{5.101}$$

holds, as expected. This was in fact previously verified in [165], where several general results regarding the behavior of $a(\theta)$ for holographic theories were discussed, including the fact that κ is not universally related to C_T, as opposed to σ. The subleading coefficients in the smooth-limit expansion are modified with respect to the Einstein gravity result in an obvious way for $\mathcal{L}^{(3)}_{(3,4,5,6,7,8)}$, but in a nontrivial one for $\mathcal{L}_1^{(3)}$ and $\mathcal{L}_2^{(3)}$. The first few of them read:

$$\sigma_{\mathcal{L}^{(3)}_{(1,2)}} = [1 + 12\beta_1 - 12\beta_2] \sigma_{\rm E} \ , \qquad \sigma'_{\mathcal{L}^{(3)}_{(1,2)}} = [1 + 15\beta_1 - 6\beta_2] \sigma'_{\rm E} \ , \tag{5.102a}$$

$$\sigma''_{\mathcal{L}^{(3)}_{(1,2)}} = \left[1 + \frac{1173}{74}\beta_1 - \frac{189}{37}\beta_2\right] \sigma''_{\rm E} \ , \qquad \sigma'''_{\mathcal{L}^{(3)}_{(1,2)}} = \left[1 + \frac{963}{65}\beta_1 - \frac{414}{65}\beta_2\right] \sigma'''_{\rm E} \ , \tag{5.102b}$$

$$\sigma^{(4)}_{\mathcal{L}^{(3)}_{(1,2)}} = \left[1 + \frac{43946}{3133}\beta_1 - \frac{24896}{3133}\beta_2\right] \sigma^{(4)}_{\rm E} \ . \tag{5.102c}$$

In the opposite limit, we find:

$$\kappa_{\rm Riem^3} = \left[1 + \frac{69}{5}\beta_1 - \frac{42}{5}\beta_2 + 6\beta_3 + 24\beta_4 + 27\beta_5 + 27\beta_6 + 108\beta_7 + 432\beta_8\right] \kappa_{\rm E} \ . \tag{5.103}$$

The coefficients for $\mathcal{L}_i^{(3)}$ with $i = 3, \ldots, 8$ are the same as those appearing in $\sigma_{\rm Riem^3}$, as they should because the corner function is proportional to the Einstein gravity one for those Lagrangians. This is not the case for $\mathcal{L}_1^{(3)}$ and $\mathcal{L}_2^{(3)}$. On the other hand, as expected on general grounds, κ_{Riem^3} matches the coefficient of the slab computed above – compare with (5.42) for $d = 3$.

Let us conclude this section by comparing the new cubic corner functions with some others previously known. For the sake of conciseness, from now on we restrict the discussion to Einsteinian Cubic Gravity, whose Lagrangian we introduced in (5.26). The corner function for this theory is given by:

$$\begin{aligned} a_{\text{ECG}}(\theta) =&(1+3\mu_{\text{ECG}})\, a_{\text{E}}(\theta) \\ &- \frac{\mu_{\text{ECG}} L_\star^2}{2G_N} \int_0^\infty \mathrm{d}y \frac{3(1+h_0^2)(15+8h_0^4(1+y^2)^2+h_0^2(23+16y^2))}{4(1+h_0^2(1+y^2))^{7/2}\sqrt{2+h_0^2(1+y^2)}} \,. \end{aligned} \tag{5.104}$$

The first smooth-limit coefficients and the sharp-limit one read in this case:

$$\sigma_{\text{ECG}} = [1-3\mu_{\text{ECG}}]\sigma_{\text{E}} \,, \qquad \sigma'_{\text{ECG}} = \left[1-\frac{33}{4}\mu_{\text{ECG}}\right]\sigma'_{\text{E}} \,, \tag{5.105a}$$

$$\sigma''_{\text{ECG}} = \left[1-\frac{2673}{296}\mu_{\text{ECG}}\right]\sigma''_{\text{E}} \,, \qquad \sigma'''_{\text{ECG}} = \left[1-\frac{2061}{260}\mu_{\text{ECG}}\right]\sigma'''_{\text{E}} \,, \tag{5.105b}$$

$$\sigma^{(4)}_{\text{ECG}} = \left[1-\frac{41023}{6266}\mu_{\text{ECG}}\right]\sigma^{(4)}_{\text{E}} \,, \qquad \kappa_{\text{ECG}} = \left[1-\frac{123}{20}\mu_{\text{ECG}}\right]\kappa_{\text{E}} \,. \tag{5.105c}$$

As shown in [149], the general bounds on the stress-tensor three-point function coefficient $-4 \leq t_4 \leq 4$ [152] impose sever constraints on the allowed values of μ_{ECG}, namely, $-0.00322 \leq \mu_{\text{ECG}} \leq 0.00312$. In the perturbative analysis we are pursuing, bounds on finite values of μ_{ECG} are not so relevant, but we can use them to give us an idea of how much it is sensible to deviate μ_{ECG} from zero when performing comparisons with other theories. In Figure 5.1, we have plotted $a_{\text{ECG}}(\theta)$ for the limiting values $\mu_{\text{ECG}} \simeq -0.00322$ and $\mu_{\text{ECG}} \simeq 0.00312$ (all intermediate values of μ_{ECG} lie between the two curves), along with the Einstein gravity result and the free scalar ($t_4 = +4$) and free fermion ($t_4 = -4$) ones [160–162]. We can see that all curves are remarkably close to each other, in agreement with the observation/conjecture of [164] that $a(\theta)/C_T$ is an almost-universal quantity for general CFTs. We observe this to be the case for the whole family of theories parametrized by the continuous parameter μ_{ECG} lying between the limiting cases extremizing the value of t_4. By making the values of $|\mu_{\text{ECG}}|$ greater, we can obtain curves which deviate more significantly from the Einstein and free-field curves (see dotted lines in Figure 5.1). However, those would correspond to toy models of CFTs which do not respect the general bounds $|t_4| \leq 4$. Hence, it is reasonable to expect that for actual CFTs the curves will indeed fall extremely close to each other in general. In fact, the ECG curves with $t_4 = 4$ and $t_4 = -4$ lie even closer to the Einstein gravity one than the scalar and fermion curves do. This suggests that the scalar field curve may be an upper bound for general CFTs.

On the other hand, the possibility that the Einstein gravity curve is a lower bound for general curves suggested in [164] seems to be ruled out by our analysis: the introduction of higher-curvature corrections allows to go below the Einstein gravity one. This is consistent with the results previously obtained in [165]. Note that such conjecture was also supported by the fact that while $t_4 = 0$ for Einstein gravity, both the scalar and the fermion curves – which have, respectively, the largest positive and negative values of t_4 allowed – lie above

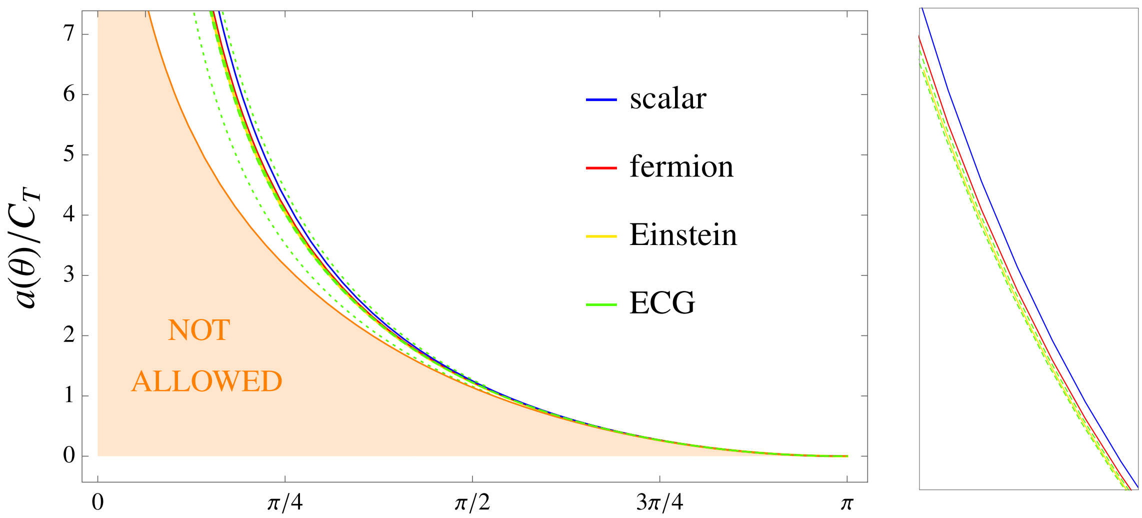

Figure 5.1: We plot the corner functions (normalized by their respective charges C_T) for a free scalar (blue), a free fermion (red), holographic Einstein gravity (yellow) and holographic Einsteinian Cubic Gravity (green). For the limit value $\mu \simeq 0.00312$ corresponding to $t_4 = +4$ (see discussion below), the curve lies very close but slightly below the Einstein gravity result (green dashed line). The case $\mu \simeq -0.00322$ corresponding to the other limit value ($t_4 = -4$) lies even closer but slightly above the Einstein gravity curve and just below the fermion one. The right plot is a zoom of the curves between $\theta = \pi/4$ and $\theta = 3\pi/8$. The orange region in the left plot is excluded for general theories by the inequality (5.77). The green dotted curves correspond to the values $\mu = -0.05$ (upper curve) and $\mu = +0.05$ (lower curve) which we have included (only) in the left plot for visual reference.

it. Here we observe that, contrary to the scalar case, ECG theories with $t_4 \geq 0$ lie below the Einstein gravity one.

In the previous subsection, we mentioned the possibility of approximating the function $a(\theta)$ for a given theory using the values of the almost-smooth and very-sharp limit coefficients, σ and κ. The proposed trial function $\tilde{a}(\theta)$ appears in (5.79). We can use the new ECG corner functions to test the accuracy of such approximation beyond the free-field and Einstein cases explored in [163]. In Figure 5.2, we plot $1 - a(\theta)/\tilde{a}(\theta)$ for various values of the ECG coupling falling between the limiting cases of $t_4 = \pm 4$. We observe that in all cases, the error in the approximation never exceeds $\sim 1.2\%$ for any value of the opening angle, the approximation being slightly better for negative values of $\mu_{\rm ECG}$. This provides good evidence that $\tilde{a}(\theta)$ can be used as an accurate approximation to the exact corner function for general CFTs.

5.5 Final discussion and conclusions

Despite being generically UV-divergent, the entanglement entropy of spatial regions in a CFT contains physically meaningful information about the field theory which can be

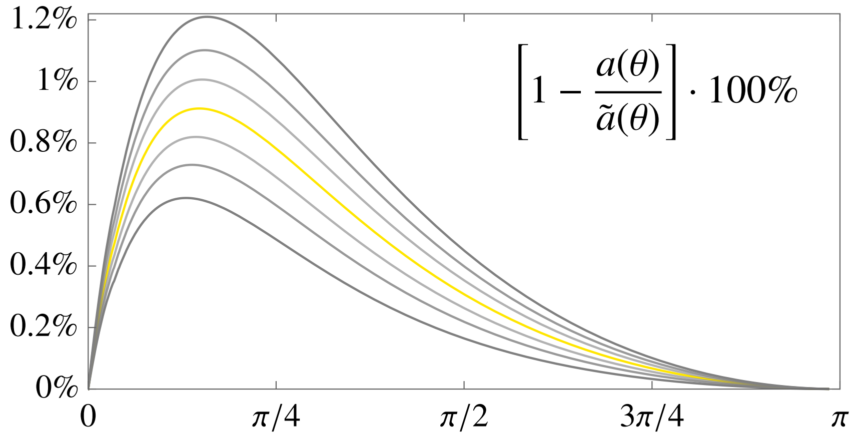

Figure 5.2: We plot $1 - a(\theta)/\tilde{a}(\theta)$ where $a(\theta)$ is the exact corner function and $\tilde{a}(\theta)$ the trial function defined in (5.79) for Einstein gravity (yellow) and ECG for different values of $\mu_{\rm ECG}$ (from top to bottom: $\mu_{\rm ECG} = +0.00312, +0.002, +0.001, -0.001, -0.002, -0.00322$). The disagreement between both functions is always smaller than $\sim 1.2\%$ throughout the whole range of values of the opening angle.

extracted if divergences are properly treated. In this chapter, we have focused on isolating universal terms for a variety of geometric shapes; these are independent of the regularization used and therefore intrinsic to the CFT. By considering the holographic entanglement entropy functional of a general (perturbative) cubic theory, we have access to different boundary CFTs, which behave differently to the one defined by bulk Einstein gravity. Thus, their universal terms provide valuable information characterizing the different theories.

In particular, spherical regions in the boundary gave us the coefficient $a^{\star(d)}$. In odd dimensions, this is related to the logarithm of the partition function of the theory in a sphere $\mathbb{S}^d$. In even dimensions, when combined with the universal terms coming from (hyper)cylindrical shapes, it is possible to compute all the coefficients appearing in the conformal anomaly, (5.6). This was explicitly done for $d = 2$ (where $a^{\star(2)}$ is directly related to the single coefficient of the conformal anomaly), $d = 4$, and $d = 6$. We also considered straight, infinite slabs in the boundary CFT. The particular symmetries of these regions give us access to a new universal coefficient, $\kappa^{(d)}$, which was caluclated for the general cubic bulk theory in any dimension. $\kappa^{(d)}$ is, up until now, not known to be related to any other CFT quantity. Therefore, it would be interesting to investigate whether field theory techniques allow to link it with some other property, as it is the case for $a^{\star(d)}$. By comparing the results for $\kappa^{(d)}$ in cubic theories with the other universal terms obtained in this chapter, it seems that this relation will not be with those more conventional quantities.

The final type of regions which were thoroughly studied were corners in $d = 3$. These are in a sense special, because they contain singular points which affect the behaviour of the entanglement entropy. The universal corner function $a(\theta)$ which appears multiplying the logarithmic divergence was introduced and, after reviewing its main proeprties, we computed its form depending on the opening angle θ for the general cubic theory. Being able to reach this order of bulk curvature corrections proved to be essential in order to obtain a different corner function from the one provided by Einstein gravity. For a particular cubic theory in the bulk, Einsteinian Cubic Gravity, we obtained the function numerically, showing that some previous conjectures could be violated for the dual CFT. In particular, the proposed lower bound for the corner function in terms of the one obtained for Einstein gravity did not hold when the cubic corrections were included. This is a good example of how higher-curvature bulk theories allow to test more general CFTs than Einstein gravity does, thus providing evidence in favour of certain conjectured general properties, while disproving others.

In summary, in the last two chapters we tried to provide a somewhat detailed picture of holographic entanglement entropy in higher-curvature gravities. Chapter 4 furnished a theoretical introduction to the subject, as well clarifying how the general functional is obtained, with a simplified procedure developed for perturbative higher-curvature corrections. The present chapter has presented one of the many applications of holographic entanglement entropy: obtaining the universal terms which characterize the CFT. This is especially relevant when higher-curvature terms are present, since these define dual field theories with different properties.

6 Summary and conclusions

Let us conclude the main text of the thesis with a short summary and some broad conclusions and open questions. We have included in each of the chapters – except the third one, which was more a concrete example of previous results than a new general discussion – detailed remarks about the main outcomes of our work, as well as some possible future directions. Those were frequently tightly linked to the technical details discussed in the corresponding chapter. We will look here for a more general perspective, trying to get the overall picture of the whole work.

The first part of the thesis considered low-energy effective gravitational actions possessing T-duality as a symmetry. These were built using Double Field Theory by Marqués and Núñez [48], and they include first-order perturbative corrections in a certain high-energy cutoff $M_\star$ – which in string theories is identified with the string length, $M_\star^{-2} \sim \alpha'$. The first obvious question is whether this construction can be extended to higher orders in the expansion. This is certainly a technically difficult task, and some recent results point towards the impossibility of obtaining by this method higher-order low-energy string effective actions, [82]. However, in some particular backgrounds such as cosmological ones, there have been interesting results, some of which allow to reach even the all-order α' expansion by imposing the $O(D, D)$ symmetry of DFT [80]. Therefore, higher-order DFT-inspired constructions are an avenue definitely worth exploring. At the very least, to show the limitations of the technique; while in a more optimistic scenario, as a way to derive useful low-energy theories with more corrections and possessing one of the defining properties of string theories: T-duality symmetry.

In case this is possible and we obtain different theories with T-duality as a built-in symmetry, we could ask similar questions to the one that motivated the first part of the thesis. T-duality is in principle just a map between solutions, since it leaves the equations of motion invariant. Whether or not those dual solutions have the same physical properties is a less clear issue. Even in the first-order family of theories discussed in the first part of the thesis, we only checked that black hole thermodynamic quantities are invariant under T-duality. More precisely, we proved the invariance of the entropy and temperature of bifurcate Killing horizons. We could think of other quantities which could behave differently under T-duality, such as for instance other asymptotic charges. It would be good to find a general argument, maybe even based on DFT constructions,[1]

[1] [172] is an interesting example along these lines for the leading order DFT action.

that guaranteed the equality of all these. The on-shell invariance of the action under T-duality implemented by construction would seem to point towards the invariance of any quantity derived from it, such as entropy or other asymptotic charges. However, we have been unable to provide a general argument along these lines.

Regarding the second part of the thesis, we have provided a useful rewriting of the holographic entanglement entropy functional in the presence of perturbative higher-curvature terms. We employed it to derive the universal terms of certain boundary regions, which give access to interesting information about the CFT dual. Studying other regions is always a straightforward possibility, although we think we have exhausted the simple cases, which are the most interesting ones in terms of analytically obtaining as much information as possible. From the CFT perspective, it would be good to know whether the universal coefficient of the slab, $\kappa^{(d)}$, is related to other interesting CFT quantities. Another avenue to further exploit the functionals obtained is to consider more general backgrounds than pure AdS. This is in general difficult, because finding solutions to the equations of motion of higher-curvature theories is not trivial at all, but in certain cases it can be done – black hole solutions are reasonably well characterized in some theories, [173].

Perhaps more interesting is the possibility of leaving the perturbative regime. At the moment, this is only possible in some special theories, such as Lovelock or $f(R)$ gravities. In general we would have to solve the equations of motion to find the right splitting, which is technically involved. However, it would be good to do it at least in some simple cases, to see whether the general rewriting in terms of differential operators is still valid. There is also an interesting alternative to go to this non-perturbative regime, which is based on the strategy followed by [120] to show that whatever the holographic entanglement entropy functional is, it has to be minimized to find the surface in which it is to be evaluated. In that work, the authors show an alternative procedure to obtain the holographic entanglement entropy functional which hides the issues posed by the splitting problem. It is still necessary to somehow deal with the equations of motion to write the final form of the functional in terms of conventional bulk quantities, but maybe the strategy can be adapted in such a way that it is easier to leave the perturbative regime than in the Lewkowycz-Maldacena construction, which requires solving the equations of motion for a general squashed cone. In any case, the approach is interesting, and properly understanding the connection with the more conventional one presented in this thesis in the context of higher-curvature gravities seems a worthwile task.

Finally, we would like to conclude with some broad reflection. In this thesis, we have used two different tools to compute entropies of higher-curvature gravities: Wald's construction based on a Noether charge originating from diffeomorphism invariance, and the holographic Lewkowycz-Maldacena method. The results they provide for stationary black holes are equivalent, since stationary Killing horizons have vanishing extrinsic curvature. This is the context in which Wald's construction can be applied, because it requires the existence of a Killing field generating the horizon. In this sense, the holographic proposal seems to be more general, as long as we could imagine applying it to non-stationary situations. This is non-trivial on its own: in Einstein gravity the generalization of the Ryu-Takayanagi proposal has to be done with some care to uplift it to a covariant version valid in cases where we do not have time translation invariance [174, 175]. However, we could imagine doing something similar with the higher-curvature functional. If possible, the resulting procedure could be applied not only to dynamical situations in holography – which is interesting on its own –, but also to the time evolution of black holes. This could shed light on discussions regarding the validity of the second law for higher-curvature theories of gravity. Thus, although this is a long term program which can find many unexpected obstacles – especially technical ones, which are always present when dealing with higher-curvature theories –, it could also be potentially very fruitful.

Supplementary material

A First order corrections to the generalized BdR entropy

This appendix will provide the details of the computation that led to the full entropy formula in the generalized BdR theory, (2.31). The leading order part, S_0, was already obtained in the main text, (2.26), so we will concentrate in the first order corrections. In splitting the action as in (2.18), there is actually a leading order piece in $\mathcal{I}_{H'^2}$, consequence of the definition of $H'_{MNR} = H_{MNR} + \mathcal{O}(a_\pm)$. Therefore, we are being somewhat sloppy by calling the whole contribution of this term a first order contribution. We will see that, in fact, the term $H_{MNR}H^{MNR}$ does not have any contribution to the leading order entropy apart from the one which goes into what we called S_{α_ξ} in (2.30c). This special part is more easily treated on its own, without splitting the leading order part from the first order correction. That is the reason to leave the analysis of the whole contribution from $\mathcal{I}_{H'^2}$ to this appendix, together with the remaining, properly called first order terms.

Let us start then by analyzing the contribution of $\mathcal{I}_{H'^2}$. First of all, we need the following result for the general variation of a Chern-Simons form built out of a connection Ω:

$$\delta\Theta = \frac{1}{3} R^A{}_B \wedge \delta\Omega^B{}_A - \frac{1}{6}\mathrm{d}\left(\Omega^A{}_B \wedge \delta\Omega^B{}_A\right) \ . \tag{A.1}$$

This is valid for any Lorentz connection, with or without torsion, and $R^A{}_B$ is the curvature 2-form associated with Ω. In particular, the functional form of $\delta\Theta^{(\pm)}$ is exactly the same just including the appropriate superscripts $(\pm)$. The previous result allows us to write the variation of H' after some algebraic manipulations as follows:

$$\begin{aligned}\delta H' = {} & \mathrm{d}\delta B - 2\gamma_- R_A{}^B \wedge \delta\Omega_B{}^A - \gamma_+ R_A{}^B \wedge \delta H_B{}^A - \gamma_+ \mathrm{d}H_A{}^B \wedge \delta\Omega_B{}^A \\ & - \frac{1}{2}\gamma_- \mathrm{d}H_A{}^B \wedge \delta H_B{}^A - 2\gamma_+ \Omega_A{}^C \wedge H_C{}^B \wedge \delta\Omega_B{}^A - \gamma_- \Omega_A{}^C \wedge H_C{}^B \wedge \delta H_B{}^A \\ & - \frac{1}{2}\gamma_- H_A{}^C \wedge H_C{}^B \wedge \delta\Omega_B{}^A - \frac{1}{4}\gamma_+ H_A{}^C \wedge H_C{}^B \wedge \delta H_B{}^A \\ & + \mathrm{d}\left[\gamma_- \Omega_A{}^B \wedge \delta\Omega_B{}^A + \frac{\gamma_-}{4} H_A{}^B \wedge \delta H_B{}^A + \frac{\gamma_+}{2}\Omega_A{}^B \wedge \delta H_B{}^A + \frac{\gamma_+}{2} H_A{}^B \wedge \delta\Omega_B{}^A\right] \ .\end{aligned} \tag{A.2}$$

Two further results are needed in order to write down the general variation of our Lagrangian. The first follows from the Hodge dual definition,

$$\delta \star H' = \frac{1}{2} G^{MN}\delta G_{MN} \star H' + \star\delta H' \ , \tag{A.3}$$

and the second is the identity $\star F \wedge G = \star G \wedge F$ for any pair of p-forms F and G. Then, we obtain the full variation of $\mathbf{L}_{H'^2} = -\frac{1}{2} e^{-2\Phi} \star H' \wedge H'$ as

$$\delta \mathbf{L}_{H'^2} = e^{-2\Phi} \left[\delta\Phi - \frac{1}{4} G^{MN} \delta G_{MN} \right] \star H' \wedge H' - e^{-2\Phi} \star H' \wedge \delta H' \ , \tag{A.4}$$

where $\delta H'$ is given by (A.2). Notice that the first term in the previous equation is going to contribute to the equations of motion without any further integration by parts and, therefore, $\boldsymbol{\theta}_{H'^2}(\Psi, \delta\Psi)$ will be obtained completely from the second term – albeit not all of it is part of the boundary term, since it also contains contributions to the equations of motion. Now, there is an obstacle to apply the derivative counting argument presented in the main text when deriving the contribution of $\mathcal{I}_0$ to the entropy. Recall that we only care about terms having one derivative of ξ in the charge $\mathbf{Q}_\xi$, which is obtained after two integrations by parts from the variation of the Lagrangian. Thus, we look for terms that would have three derivatives of the vector field ξ when we evaluate $\delta_\xi \mathbf{L}$. When considering a general variation as in (2.20), this argument is correct for the part of the transformations depending explicitly on ζ, since at the end of the calculation we are going to set $\zeta = \xi$ and evaluate at the bifurcation surface. It is also valid for the contribution proportional to λ, since we will evaluate for $\lambda = \lambda^E_\xi$, which is defined in (2.12) and contains a single derivative of ξ. But we cannot proceed in the same way with the gauge term $\mathrm{d}\beta$ appearing for the B field. As a consequence, we will derive first the contributions to the entropy charge arising from ζ and λ, leaving that of β for later analysis.[1]

Suppose then for a moment that we are working with the symmetry transformations (2.20) without $\mathrm{d}\beta$:

$$\delta_{\zeta,\lambda} \Phi = \mathcal{L}_\zeta \Phi \ , \tag{A.5a}$$

$$\delta_{\zeta,\lambda} E^A = \mathcal{L}_\zeta E^A + E^B \lambda_B{}^A \ , \tag{A.5b}$$

$$\delta_{\zeta,\lambda} B = \mathcal{L}_\zeta B + \gamma_- \Omega_A{}^B \wedge \mathrm{d}\lambda_B{}^A + \frac{\gamma_+}{2} H_A{}^B \wedge \mathrm{d}\lambda_B{}^A \ . \tag{A.5c}$$

As we said, all the contribution to the boundary term comes from the last part of (A.4), and since $\delta H'$ is given by (A.2), we can start our derivative counting process. First of all, in (2.21) we provided $\delta_{\zeta,\lambda} H_A{}^B$ just to leading order,

$$\delta_{\zeta,\lambda} H_A{}^B = \mathcal{L}_\zeta H_A{}^B - \lambda_A{}^C H_C{}^B + H_A{}^C \lambda_C{}^B + \mathcal{O}(\gamma_\pm) \ . \tag{A.6}$$

This is enough given the form of (A.2); δH it is always multiplied by γ_+ or γ_-. Since $\lambda_A{}^B$ will have at most one derivative of the vector field when evaluated on $\lambda = \lambda^E_\xi$, its differential appearing in $\delta_{\zeta,\lambda} B$ and $\delta_{\zeta,\lambda} \Omega_A{}^B$ will have two derivatives. It is then easy to find the only terms containing three derivatives in $\delta \mathbf{L}_{H'^2}$. After an integration by parts, these produce the following relevant part of the boundary term:

$$\boldsymbol{\theta}_{H'^2}(\Psi, \delta\Psi) = (-1)^D e^{-2\Phi} \left[\star H' \wedge \delta B + \star H \wedge \left(\gamma_- \Omega_A{}^B + \frac{\gamma_+}{2} H_A{}^B \right) \wedge \delta\Omega_B{}^A \right] + \dots \ , \tag{A.7}$$

where we used the fact that $H' = H + \mathcal{O}(\gamma_\pm)$. Now, in the current only terms containing two derivatives of ζ are relevant,

$$\mathbf{j}_{H'^2,\zeta,\lambda} = (-1)^D e^{-2\Phi} \star H \wedge \left[2\gamma_- \Omega_A{}^B + \gamma_+ H_A{}^B \right] \wedge \mathrm{d}\lambda_B{}^A + \dots \ , \tag{A.8}$$

[1] To be as clear as possible with the following calculations, we write explicitly the parameters of the transformation we are considering instead of Γ. Parameters taken to be zero are not written.

and another integration by parts leads us to the charge presented in the main text,

$$\mathbf{Q}_{H'^2,\zeta,\lambda} = e^{-2\Phi} \star H \wedge \left(2\gamma_- \Omega_A{}^B + \gamma_+ H_A{}^B\right) \lambda_B{}^A + \dots \ , \tag{A.9}$$

after we take $\zeta = \xi$ and $\lambda = \lambda^E_\xi$, see (2.28).

Similar calculations to the ones just presented allow us to obtain the contribution of $\mathcal{I}_{R^2}$; again, if we do not consider the gauge transformation term $\mathrm{d}\beta$ as in (A.5). First of all, the variation of the Lagrangian is given by:

$$\begin{aligned}\delta \mathbf{L}_{R^2} = \sum_{k=\pm} \frac{a_k}{4} e^{-2\Phi} \Bigg[\left(-\delta\Phi + \frac{1}{4} G^{MN} \delta G_{MN}\right) (\star R_A^{(k)B} \wedge R_B^{(k)A}) \\ +2 \star R_A^{(k)B} \wedge \delta R_B^{(k)A} \Bigg] \ ,\end{aligned} \tag{A.10}$$

where:

$$\delta R_A^{(k)B} = \mathrm{d}\left(\delta \Omega_A^{(k)B}\right) + \delta\left(\Omega_A^{(k)C} \wedge \Omega_C^{(k)B}\right) \ . \tag{A.11}$$

Note that now all the relevant contribution to $\boldsymbol{\theta}_{R^2}(\Psi, \delta\Psi)$ will come from the first term containing the differential of $\delta\Omega_A^{(k)B}$. It takes a simple calculation to conclude that:

$$\boldsymbol{\theta}_{R^2}(\Psi, \delta\Psi) = (-1)^D \sum_{k=\pm} \frac{a_k}{2} e^{-2\Phi} \star R_A^{(k)B} \wedge \delta\Omega_B^{(k)A} + \dots \tag{A.12}$$

We can rewrite this expression in terms of the parameters $\gamma_\pm$ as

$$\begin{aligned}\boldsymbol{\theta}_{R^2}(\Psi, \delta\Psi) = -\frac{(-1)^D}{2} e^{-2\Phi} \Bigg\{ \star \Bigg[4\gamma_+ \left(R_A{}^B + \frac{1}{4} H_A{}^C \wedge H_C{}^B\right) \\ +2\gamma_- \left(\mathrm{d}H_A{}^B + 2\Omega_A{}^C \wedge H_C{}^B\right)\Bigg] \wedge \delta\Omega_B{}^A \Bigg\} + \dots\end{aligned} \tag{A.13}$$

The current is now given by:

$$\begin{aligned}\mathbf{j}_{R^2,\zeta,\lambda} = -(-1)^D e^{-2\Phi} \Bigg\{ 2\gamma_+ \star \left(R_A{}^B + \frac{1}{4} H_A{}^C \wedge H_C{}^B\right) \wedge \mathrm{d}\lambda_B{}^A \\ + \gamma_- \star \left(\mathrm{d}H_A{}^B + 2\Omega_A{}^C \wedge H_C{}^B\right) \wedge \mathrm{d}\lambda_B{}^A \Bigg\} + \dots \ ,\end{aligned} \tag{A.14}$$

and the corresponding charge for the entropy would be

$$\begin{aligned}\mathbf{Q}_{R^2,\zeta,\lambda} = -e^{-2\Phi} \Bigg[2\gamma_+ \star \left(R_A{}^B + \frac{1}{4} H_A{}^C \wedge H_C{}^B\right) \\ +\gamma_- \star \left(\mathrm{d}H_A{}^B + 2\Omega_A{}^C \wedge H_C{}^B\right)\Bigg] \lambda_B{}^A + \dots\end{aligned} \tag{A.15}$$

This is the result in (2.28) (taking $\zeta = \xi$ and $\lambda = \lambda^E_\xi$), but it is puzzling at first sight. We seem to have a γ_- contribution to the entropy, but appendix B of [48] shows that the action $\mathcal{I}_{R^2}$ has no γ_- part. Their proof relies upon Bianchi identities, and using them we can also conclude that the γ_- part of the entropy vanishes. Let us sketch the proof as follows. First of all, we can use the antisymmetry of ${\lambda_B}^A$ to rewrite:

$$\star\left(\mathrm{d}{H_A}^B + 2{\Omega_A}^C\wedge {H_C}^B\right){\lambda_B}^A = \star\left(\mathrm{d}{H_A}^B + {\Omega_A}^C\wedge {H_C}^B + {H_A}^C\wedge{\Omega_C}^B\right){\lambda_B}^A\ . \tag{A.16}$$

This is a Lorentz covariant derivative for H,

$$\begin{aligned} Y^{AB} &\equiv \mathrm{d}H^{AB} + \Omega^{AC}\wedge {H_C}^B + H^{AC}\wedge {\Omega_C}^B \\ &= E^{AM}E^{BN}\nabla_R H_{MNS}\,\mathrm{d}x^R\wedge\mathrm{d}x^S\ . \end{aligned} \tag{A.17}$$

This expression, when evaluated on $\mathcal{B}$, will be contracted with the binormal n_{AB}, since for $\lambda = \lambda^E_\xi$ we know that $(\lambda^E_\xi)_{BA}|_{\mathcal{B}} = n_{AB}$, [85].[2] Besides, taking also the Hodge dual we obtain:

$$\star\left(Y^{AB}n_{AB}\right) = \star\left(n^{MN}\nabla_R H_{MNS}\,\mathrm{d}x^R\wedge\mathrm{d}x^S\right) = 2n^{MN}\nabla^R {H^S}_{MN}\left(\mathrm{d}^{D-2}x\right)_{RS}\ . \tag{A.18}$$

Using now $\left(\mathrm{d}^{D-1}x\right)_{RS}|_{\mathcal{B}} = n_{RS}\,\bar{\boldsymbol{\epsilon}}/2$, we can show, as a consequence of $\mathrm{d}H = 0$, that:

$$n^{MN}n^{RS}\nabla_M H_{NRS} = \frac{1}{2}n^{MN}n^{RS}\left[\nabla_{[M}H_{N]RS} + \nabla_{[R}H_{S]MN}\right] = 0\ . \tag{A.19}$$

So the γ_- terms in $\mathbf{Q}_{R^2,\zeta,\lambda}$ vanish as they should. We finally obtain:

$$S_{R^2} = -4\pi\gamma_+\int_{\mathcal{B}} e^{-2\Phi}\star\left(R^{AB} + \frac{1}{4}H^{AC}\wedge {H_C}^B\right)n_{AB}\ , \tag{A.20}$$

which is the expression for the entropy presented in (2.30b).

Let us now come back to the issue of the gauge symmetry of the B-field parametrized by β. The first thing we have to realize is that these kind of gauge contributions to the entropy charge arise when considering both $\mathcal{I}_{H'^2}$ and $\mathcal{I}_{R^2}$. It will prove to be a good idea to tackle the full problem all at once, instead of isolating the two separate pieces. Consider then our full Lagrangian form $\mathbf{L} = \mathcal{L}\,\boldsymbol{\epsilon}$, which depends on B_{MN} only through H_{MNR} and its first derivatives; the latter appearing from $R^{(\pm)B}_{MNA}$ and $\Theta^{(\pm)}_{MNR}$. From a general variation just involving the B-field, it is easy to obtain:

$$\begin{aligned} \delta_B\mathbf{L} &= \boldsymbol{\epsilon}\left[T^{MNR}\delta_B H_{MNR} + S^{QMNR}\delta_B\nabla_Q H_{MNR}\right] \\ &= -3\boldsymbol{\epsilon}\nabla_M\mathbb{E}^{MNR}\delta B_{NR} + 3\boldsymbol{\epsilon}\nabla_M\left[\mathbb{E}^{MNR}\delta B_{NR} + S^{M[QNR]}\nabla_Q\delta B_{NR}\right]\ , \end{aligned} \tag{A.21}$$

where we have made use of the definitions (2.29):

$$T^{MNR} \equiv \frac{\partial\mathcal{L}}{\partial H_{MNR}}\ ,\quad S^{QMNR} \equiv \frac{\partial\mathcal{L}}{\partial\nabla_Q H_{MNR}}\ ,\quad \mathbb{E}^{MNR} \equiv T^{MNR} - \nabla_Q S^{QMNR}\ . \tag{A.22}$$

The Euler-Lagrange equation for the B-field has the form:

$$\nabla_M\mathbb{E}^{MNR} \cong 0\ , \tag{A.23}$$

[2] Recall that, for the entropy computation, we consider the Killing field to be normalized as $\nabla_M\xi_N|_{\mathcal{B}} = n_{MN}$, as required by (2.17).

whereas the boundary term is just

$$\theta^M(\Psi, \delta B) = 3\mathbb{E}^{MNR}\delta B_{NR} + S^{MQNR}\delta H_{QNR} \ . \tag{A.24}$$

We can now easily obtain the contribution from the gauge parameter β of the symmetry transformations (2.20), that we denoted by δ_β. Clearly, $\delta_\beta H_{MNR} = 0$, and thus the contributions to the current and charge, proportional to β, will be:

$$j^M_{B,\beta} = 6\mathbb{E}^{MNR}\nabla_N \beta_R \ , \qquad Q^{MN}_{B,\beta} = 6\mathbb{E}^{MNR}\beta_R \ , \tag{A.25}$$

where we employed the fact that $\nabla_N \mathbb{E}^{MNR} \cong 0$. This is the charge presented in the main text if we define $\mathbf{Q}_{\alpha_\xi} := \mathbf{Q}_{B,\alpha_\xi}$. As a byproduct of this result, we can also conclude that the addition to α_ξ of an exact form will not change the entropy value, since taking $\alpha_\xi = \mathrm{d}\gamma$ we can write $\mathbf{Q}_{\alpha_\xi}$ as a total derivative to be integrated over the bifurcation surface, which we assume has no boundary, as in [91].

B

Proof of the rewriting of the HEE functional

Let us present here the detailed argument that shows how to obtain the derivative form of the anomaly term, (4.49), from the original form proposed by [52] and presented in the main text in (4.21). The α expansion appearing in that expression is performed on

$$\left(\frac{8\partial^2\mathcal{L}_E}{\partial \text{Riem}^2} K^2\right) \equiv 8 \frac{\partial^2\mathcal{L}_E}{\partial R_{zizj}\partial R_{\bar{z}k\bar{z}l}} K_{zij} K_{\bar{z}kl} \; . \tag{B.1}$$

This object is a complicated combination of the different basic Riemann tensor components, $R_{\mathcal{M}I}$ (see the discussion before (4.23) for the different explicit terms this represents). The α expansion instructus us to split each component schematically as

$$R_{\mathcal{M}I} = \tilde{R}_{\mathcal{M}I} + \mathcal{K}_{\mathcal{M}I} \; , \tag{B.2}$$

where $\tilde{R}_{\mathcal{M}I}$ has $q_\alpha = 0$ and $\mathcal{K}_{\mathcal{M}I}$ a certain non-zero value of q_α. We have to isolate each monomial in (B.1), compute its total q_α value by summing over its constituents, and finally divide by $(1+q_\alpha)$. We will replicate these steps in an abstract way making use of the properties of derivatives and Taylor-like expansions.

It is illustrative to consider first a simplified version of the problem to better understand the strategy. Suppose we have some function $f(x)$ and we want to evaluate it at $\tilde{x}+k$ in a way such that we explicitly isolate monomials depending on the number of k factors they have. A simple way to do this is by Taylor-expanding $f(\tilde{x}+k)$ around a certain x, namely,

$$f(\tilde{x}+k) = \sum_{S=0}^{\infty} \frac{1}{S!} \left(\tilde{x}+k-x\right)^S \partial_x^S f(x) \; , \tag{B.3}$$

and then using the binomial theorem,

$$f(\tilde{x}+k) = \sum_{S=0}^{\infty} \sum_{\lambda=0}^{S} \frac{1}{\lambda!(S-\lambda)!} \left(\tilde{x}-x\right)^{S-\lambda} k^\lambda \partial_x^S f(x) \; . \tag{B.4}$$

Notice that this expression does not depend on x, as the left-hand side explicitly indicates: it is only an arbitrary point of expansion. Furthermore, the number of k's in each term is equal to λ, so in case we want to divide by a factor depending on the number of k's, it is easy to do term by term. This is exactly what we will want to do in the anomaly piece of the holographic entanglement entropy functional. Before writing the equivalent expression there, let us introduce an extra piece of notation that will make our life easier later. Notice that we can pair each of the S derivatives with the factors $(\tilde{x}-x)$ and k provided we introduce some ordering convention. The idea is to impose that derivatives only act on $f(x)$ and not on explicit x factors, and we denote this by enclosing our expression in between $::$, a notation reminiscent of normal ordering:

$$f(\tilde{x}+k)=\left[:\sum_{S=0}^{\infty}\sum_{\lambda=0}^{S}\frac{1}{\lambda!(S-\lambda)!}\left[(\tilde{x}-x)\,\partial_x\right]^{S-\lambda}(k\partial_x)^{\lambda}:\right]f(x)\,. \tag{B.5}$$

Now, with some care, the idea presented above can be extended to functions of several variables. In the case of interest here, these variables will be Riemann tensor components. Roughly speaking, $f(x)$ will be replaced by the object defined in (B.1), and evaluating at $x\to\tilde{x}+k$ will be the splitting of each component, $R_{\mathcal{M}I}\to\tilde{R}_{\mathcal{M}I}+\mathcal{K}_{\mathcal{M}I}$. Expanding around $R_{\mathcal{M}I}$,

$$\left.\left(\frac{8\partial^2\mathcal{L}_E}{\partial\text{Riem}^2}K^2\right)\right|_{\tilde{R}+\mathcal{K}}=\sum_{S=0}^{\infty}\frac{1}{S!}\left(\tilde{R}_{\mathcal{M}_1I_1}+\mathcal{K}_{\mathcal{M}_1I_1}-R_{\mathcal{M}_1I_1}\right)\dots\left(\tilde{R}_{\mathcal{M}_nI_n}+\mathcal{K}_{\mathcal{M}_nI_n}-R_{\mathcal{M}_nI_n}\right)$$
$$\times\frac{\hat{\partial}}{\hat{\partial}R_{\mathcal{M}_1I_1}\dots\hat{\partial}R_{\mathcal{M}_nI_n}}\left(\frac{8\partial^2\mathcal{L}_E}{\partial\text{Riem}^2}K^2\right)\,. \tag{B.6}$$

Notice the use of the hatted derivatives defined in (4.45): as argued there, these are the correct ones when doing Taylor expansions, for they properly take into account the symmetries of the different Riemann tensor components. Notice also the advantage of the normal ordering prescription, since using it we can rewrite the previous expression as:

$$\left.\left(\frac{8\partial^2\mathcal{L}_E}{\partial\text{Riem}^2}K^2\right)\right|_{\tilde{R}+\mathcal{K}}$$
$$=\left[:\sum_{S=0}^{\infty}\frac{1}{S!}\left(\left(\tilde{R}_{\mathcal{M}I}+\mathcal{K}_{\mathcal{M}I}-R_{\mathcal{M}I}\right)\hat{\partial}^{\mathcal{M}I}\right)^S:\right]\left(\frac{8\partial^2\mathcal{L}_E}{\partial\text{Riem}^2}K^2\right)\,, \tag{B.7}$$

where the precise meaning of $::$ is that derivatives only act on the object completely to the right of the expression,

$$:\left(R_{\mathcal{M}I}\hat{\partial}^{\mathcal{M}I}\right)^n:\equiv R_{\mathcal{M}_1I_1}\dots R_{\mathcal{M}_nI_n}\frac{\hat{\partial}}{\hat{\partial}R_{\mathcal{M}_1I_1}\dots\hat{\partial}R_{\mathcal{M}_nI_n}}\,. \tag{B.8}$$

This makes the object inside the S power in (B.7) a commuting one, because derivatives are to be moved to the right before applying them. From now on, we work with the operator between brackets alone, since it contains all we need, namely, the explicit dependence on the $\mathcal{K}_{\mathcal{M}I}$. To avoid clutter, we will also implicitly assume in this appendix the normal ordering convention for derivatives.

From now on, we will have to deal separately with the two types of Riemann tensor components: those we called type-A (with $q_\alpha=1$ for the corresponding $\mathcal{K}_{\mathcal{M}I}$) and those we called type-B (with $q_\alpha=1/2$ for the corresponding $\mathcal{K}_{\mathcal{M}I}$). This can be easily done from the previous expression,

$$\sum_{S=0}^{\infty}\frac{1}{S!}\left(\left(\tilde{R}_{\mathcal{A}I}+\mathcal{K}_{\mathcal{A}I}-R_{\mathcal{A}I}\right)\hat{\partial}^{\mathcal{A}I}+\left(\tilde{R}_{\mathcal{B}I}+\mathcal{K}_{\mathcal{B}I}-R_{\mathcal{B}I}\right)\hat{\partial}^{\mathcal{B}I}\right)^{S}= \tag{B.9}$$
$$=\sum_{S=0}^{\infty}\sum_{T=0}^{S}\frac{1}{T!(S-T)!}\left[\left(\tilde{R}_{\mathcal{A}I}+\mathcal{K}_{\mathcal{A}I}-R_{\mathcal{A}I}\right)\hat{\partial}^{\mathcal{A}I}\right]^{T}\left[\left(\tilde{R}_{\mathcal{B}J}+\mathcal{K}_{\mathcal{B}J}-R_{\mathcal{B}J}\right)\hat{\partial}^{\mathcal{B}J}\right]^{S-T},$$

where we used the binomial theorem, for which it is essentiall that the objects inside the S power commute. The next step is to isolate the number of $\mathcal{K}$'s of each type, to prepare for the $(1+q_\alpha)$ division,

$$\sum_{S=0}^{\infty}\sum_{T=0}^{S}\sum_{\lambda_1=0}^{T}\sum_{\lambda_2=0}^{S-T}\frac{1}{T!(S-T)!}\frac{T!}{\lambda_1!(T-\lambda_1)!}\left[\mathcal{K}_{\mathcal{A}I}\hat{\partial}^{\mathcal{A}I}\right]^{\lambda_1}\left[\left(\tilde{R}_{\mathcal{A}'I'}-R_{\mathcal{A}'I'}\right)\hat{\partial}^{\mathcal{A}'I'}\right]^{T-\lambda_1}$$
$$\frac{(S-T)!}{\lambda_2!(S-T-\lambda_2)!}\left[\mathcal{K}_{\mathcal{B}J}\hat{\partial}^{\mathcal{B}J}\right]^{\lambda_2}\left[\left(\tilde{R}_{\mathcal{B}'J'}-R_{\mathcal{B}'J'}\right)\hat{\partial}^{\mathcal{B}'J'}\right]^{S-T-\lambda_2}. \tag{B.10}$$

In this expression, it is manifest that we have λ_1 components $\mathcal{K}_{\mathcal{A}I}$, which contribute 1 to q_α, and λ_2 components $\mathcal{K}_{\mathcal{B}J}$, which contribute $1/2$ to q_α. Hence, we are ready to divide by $(1+q_\alpha)=(1+\lambda_1+\lambda_2/2)$, obtaining

$$\sum_{S=0}^{\infty}\sum_{T=0}^{S}\sum_{\lambda_1=0}^{T}\sum_{\lambda_2=0}^{S-T}\frac{2}{(2+2\lambda_1+\lambda_2)}\frac{1}{\lambda_1!(T-\lambda_1)!}\left[\mathcal{K}_{\mathcal{A}I}\hat{\partial}^{\mathcal{A}I}\right]^{\lambda_1}\left[\left(\tilde{R}_{\mathcal{A}'I'}-R_{\mathcal{A}'I'}\right)\hat{\partial}^{\mathcal{A}'I'}\right]^{T-\lambda_1}$$
$$\frac{1}{\lambda_2!(S-T-\lambda_2)!}\left[\mathcal{K}_{\mathcal{B}J}\hat{\partial}^{\mathcal{B}J}\right]^{\lambda_2}\left[\left(\tilde{R}_{\mathcal{B}'J'}-R_{\mathcal{B}'J'}\right)\hat{\partial}^{\mathcal{B}'J'}\right]^{S-T-\lambda_2}. \tag{B.11}$$

At this point, the α-sum has been performed, and we do not need to explicitly keep the $\mathcal{K}$ dependence isolated. We can also rewrite the $\tilde{R}$ back in terms of conventional Riemann tensor components. Using $\tilde{R}_{\mathcal{M}I}=R_{\mathcal{M}I}-\mathcal{K}_{\mathcal{M}I}$ we have:

$$\sum_{S=0}^{\infty}\sum_{T=0}^{S}\sum_{\lambda_1=0}^{T}\sum_{\lambda_2=0}^{S-T}\frac{2}{(2+2\lambda_1+\lambda_2)}\frac{(-1)^{S-\lambda_1-\lambda_2}}{\lambda_1!(T-\lambda_1)!}$$
$$\times\frac{1}{\lambda_2!(S-T-\lambda_2)!}\left(\mathcal{K}_{\mathcal{A}I}\hat{\partial}^{\mathcal{A}I}\right)^{T}\left(\mathcal{K}_{\mathcal{B}J}\hat{\partial}^{\mathcal{B}J}\right)^{S-T}. \tag{B.12}$$

At this point we proceed to perform the λ_1 and λ_2 sums, which do not affect the derivative operators. Let us start with the λ_2 one. We have that:

$$\sum_{\lambda_2=0}^{S-T}\frac{(-1)^{\lambda_2}}{(2+2\lambda_1+\lambda_2)}\frac{1}{\lambda_2!(S-T-\lambda_2)!}=\frac{(2\lambda_1+1)!}{(2\lambda_1+2+S-T)!}. \tag{B.13}$$

To show this, take $n \equiv 2 + 2\lambda_1$ and $\tilde{T} \equiv S - T$, and manipulate as follows:

$$\begin{aligned}
\sum_{\lambda=0}^{\tilde{T}} \frac{(-1)^\lambda}{(\lambda+n)} \frac{1}{\lambda!(\tilde{T}-\lambda)!} &= \sum_{\lambda=0}^{\tilde{T}} \frac{(-1)^\lambda}{(\lambda+n)!(\tilde{T}-\lambda)!}(\lambda+n-1)\cdots(\lambda+1) \\
&= \frac{1}{(\tilde{T}+n)!}\partial_x^{n-1}\left[\sum_{\lambda=0}^{\tilde{T}} \binom{\tilde{T}+n}{\lambda+n}(-1)^\lambda x^{\lambda+n-1}\right]_{x=1} \\
&= \frac{1}{(\tilde{T}+n)!}\partial_x^{n-1}\left[\sum_{\lambda=n}^{\tilde{T}+n} \binom{\tilde{T}+n}{\lambda}(-1)^{\lambda-n} x^{\lambda-1}\right]_{x=1} \\
&= \frac{1}{(\tilde{T}+n)!}\partial_x^{n-1}\left[\sum_{\lambda=n}^{\tilde{T}+n} \binom{\tilde{T}+n}{\lambda}(-1)^{\lambda-n} x^{\lambda-1}\right]_{x=1} \\
&= \frac{(-1)^n}{(\tilde{T}+n)!}\partial_x^{n-1}\left[\frac{1}{x}\sum_{\lambda=0}^{\tilde{T}+n} \binom{\tilde{T}+n}{\lambda}(-x)^\lambda - \frac{1}{x}\binom{\tilde{T}+n}{0}\right. \\
&\qquad\qquad \left. - \cdots - \binom{\tilde{T}+n}{n-1}(-x)^{n-2}\right]_{x=1} \\
&= \frac{(-1)^n}{(\tilde{T}+n)!}\partial_x^{n-1}\left[\frac{(1-x)^{\tilde{T}+n}}{x} - \frac{1}{x}\right]_{x=1} \\
&= \frac{(n-1)!}{(\tilde{T}+n)!} \, ,
\end{aligned}$$

where the first term inside the brackets does not survive after $(n-1)$ derivatives evaluated at $x = 1$ because of the factor $(1-x)^{\tilde{T}+n}$, and we have used:

$$\partial_x^{n-1}\left(\frac{1}{x}\right) = \frac{(-1)^{n-1}(n-1)!}{x^n} \, . \tag{B.14}$$

With (B.13), the operator (B.12) becomes:

$$2\sum_{S=0}^{\infty}\sum_{T=0}^{S}\sum_{\lambda_1=0}^{T} \frac{(-1)^{S-\lambda_1}}{\lambda_1!(T-\lambda_1)!}\frac{(2\lambda_1+1)!}{(2\lambda_1+2+S-T)!}\left(\mathcal{K}_{\mathcal{A}I}\hat{\partial}^{\mathcal{A}I}\right)^T\left(\mathcal{K}_{\mathcal{B}J}\hat{\partial}^{\mathcal{B}J}\right)^{S-T} \, . \tag{B.15}$$

We can now try to do the λ_1 sum. We find the following integral representation:[1]

$$\sum_{\lambda_1=0}^{T} \frac{(-1)^{\lambda_1}}{\lambda_1!(T-\lambda_1)!}\frac{(2\lambda_1+1)!}{(2\lambda_1+2+S-T)!} = \frac{1}{T!(S-T)!}\int_0^1 \mathrm{d}u\, u(1-u^2)^T(1-u)^{S-T} \, . \tag{B.17}$$

[1]This can be explicitly written in terms of Gauss' hypergeometric function as

$$\sum_{\lambda=0}^{T} \frac{(-1)^\lambda}{\lambda!(T-\lambda)!}\frac{(2\lambda+1)!}{(2\lambda+2+S-T)!} = \frac{2+S-(S-T)\,{}_2F_1\left[1,-T;3+S;-1\right]}{2(1+S)(2+S)(S-T)!T!} \, , \tag{B.16}$$

but the integral form turns out to be more useful for our purposes.

Again, to prove this, rename $m \equiv 2 + S - T$ and proceed as follows:

$$
\begin{aligned}
\sum_{\lambda=0}^{T} \frac{(-1)^\lambda}{\lambda!(T-\lambda)!} \frac{(2\lambda+1)!}{(2\lambda+m)!} &= \sum_{\lambda=0}^{T} \frac{(-1)^\lambda}{\lambda!(T-\lambda)!} \frac{1}{(2\lambda+m)\cdots(2\lambda+2)} \\
&= \sum_{\lambda=0}^{T} \frac{(-1)^\lambda}{\lambda!(T-\lambda)!} \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{m-2}} \mathrm{d}x_{m-1} x_{m-1}^{2\lambda+1} \\
&= \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{m-2}} \mathrm{d}x_{m-1}\, x_{m-1} \sum_{\lambda=0}^{T} \frac{(-x_{m-1}^2)^\lambda}{\lambda!(T-\lambda)!} \\
&= \frac{1}{T!} \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{m-3}} \mathrm{d}x_{m-2} \int_0^{x_{m-2}} \mathrm{d}z\, z(1-z^2)^T \,,
\end{aligned}
$$

where we have relabelled $x_{m-1} \equiv u$. We can reorder the integrals now using the following identity:

$$
\begin{aligned}
&\int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{m-3}} \mathrm{d}x_{m-2} \int_0^{x_{m-2}} \mathrm{d}u\, f(u) \\
&= \int_0^1 \mathrm{d}u\, f(u) \int_u^1 \mathrm{d}x_1 \int_u^{x_1} \mathrm{d}x_2 \cdots \int_u^{x_{m-3}} \mathrm{d}x_{m-2}\,,
\end{aligned}
$$

and then use,

$$
\int_u^1 \mathrm{d}x_1 \int_u^{x_1} \mathrm{d}x_2 \cdots \int_u^{x_{m-3}} \mathrm{d}x_{m-2} = \frac{1}{(m-2)!} \left(\int_u^1 \mathrm{d}x \right)^{m-2} = \frac{(1-u)^{m-2}}{(m-2)!} \,,
$$

to finally obtain (B.17). Going back to (B.15), we can now write the operator as:

$$
\begin{aligned}
&\int_0^1 \mathrm{d}u\, 2u \sum_{S=0}^{\infty} \sum_{T=0}^{S} \frac{(-1)^S}{T!(S-T)!} \left((1-u^2)\mathcal{K}_{\mathcal{A}I} \hat{\partial}^{\mathcal{A}I} \right)^T \left((1-u)\mathcal{K}_{\mathcal{B}J} \hat{\partial}^{\mathcal{B}J} \right)^{S-T} \\
&= \sum_{S=0}^{\infty} \frac{(-1)^S}{S!} \int_0^1 \mathrm{d}u\, 2u \left[(1-u^2)\mathcal{K}_{\mathcal{A}I} \hat{\partial}^{\mathcal{A}I} + (1-u)\mathcal{K}_{\mathcal{B}J} \hat{\partial}^{\mathcal{B}J} \right]^S \,. \qquad \text{(B.18)}
\end{aligned}
$$

This is the final result presented in the main text, (4.49):

$$
\begin{aligned}
&\sum_{\alpha} \frac{1}{1+q_\alpha} \left(\frac{8\partial^2 \mathcal{L}_E}{\partial \mathrm{Riem}^2} K^2 \right)_\alpha \qquad \text{(B.19)} \\
&= \sum_{S=0}^{\infty} \frac{1}{S!} \int_0^1 \mathrm{d}u\, 2u : \left[-(1-u^2)\mathcal{K}_{\mathcal{A}I} \hat{\partial}^{\mathcal{A}I} - (1-u)\mathcal{K}_{\mathcal{B}J} \hat{\partial}^{\mathcal{B}J} \right]^S : \left(\frac{8\partial^2 \mathcal{L}_E}{\partial \mathrm{Riem}^2} K^2 \right) ,
\end{aligned}
$$

where, in this expression, $\mathcal{K}_{\mathcal{A}I} \hat{\partial}^{\mathcal{A}I}$ and $\mathcal{K}_{\mathcal{B}J} \hat{\partial}^{\mathcal{B}J}$ are the ones given in (4.50).

C

Cubic and quartic HEE functionals

This is an appendix collecting the results for cubic and quartic holographic entanglement entropy functionals. Since the expressions are quite long, and not particularly illuminating, we found convinient to relegate them to this auxiliary section. Starting with the general cubic theory:

$$I_E^{\mathrm{Riem}^3} = -\frac{1}{16\pi G_N}\int \mathrm{d}^{d+1}x\sqrt{G}\left[\frac{d(d-1)}{L^2} + R + L^4\sum_{i=1}^{8}\beta_i\mathcal{L}_i^{(3)}\right] , \tag{C.1}$$

with

$$\mathcal{L}_1^{(3)} \equiv R_{M\ N}^{\ R\ S}R_{R\ S}^{\ P\ Q}R_{P\ Q}^{\ M\ N} , \qquad \mathcal{L}_2^{(3)} \equiv R_{MN}^{\ \ \ RS}R_{RS}^{\ \ \ PQ}R_{PQ}^{\ \ \ MN} , \tag{C.2a}$$

$$\mathcal{L}_3^{(3)} \equiv R_{MNRS}R^{MNR}{}_{P}R^{SP} , \qquad \mathcal{L}_4^{(3)} \equiv R_{MNRS}R^{MNRS}R , \tag{C.2b}$$

$$\mathcal{L}_5^{(3)} \equiv R_{MNRS}R^{MR}R^{NS} , \qquad \mathcal{L}_6^{(3)} \equiv R_M^N R_N^R R_R^M , \tag{C.2c}$$

$$\mathcal{L}_7^{(3)} \equiv R_{MN}R^{MN}R , \qquad \mathcal{L}_8^{(3)} \equiv R^3 . \tag{C.2d}$$

The corresponding functional has the form:

$$S_{\mathrm{EE}}^{\mathrm{Riem}^3} = \frac{\mathrm{Area}(\Gamma_A)}{4G_N} + \frac{L^4}{4G_N}\int_{\Gamma_A}\mathrm{d}^{d-1}y\sqrt{h}\sum_{i=1}^{8}\beta_i\Delta_i^{(3)} + \mathcal{O}(\beta_i^2) . \tag{C.3}$$

with

$$\begin{aligned}
\Delta_1^{(3)} =& + \frac{3}{2}\left(R^{aN}{}_{aM}R^{bM}{}_{bN} - R^{aNbM}R_{aMbN}\right) && \text{(C.4a)}\\
& - \frac{3}{2}R^{ijkl}K_{aik}K^{a}{}_{jl} - 3R^{abij}K_{ai}{}^{k}K_{bjk} + \frac{3}{4}R^{ab}{}_{ab}K^{cij}K_{cij} - \frac{3}{8}K^{aij}K_{aij}K^{bkl}K_{bkl}\\
& + \frac{9}{4}K_{ai}{}^{j}K_{bj}{}^{k}K^{a}{}_{k}{}^{l}K^{b}{}_{l}{}^{i} - \frac{3}{2}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K^{b}{}_{l}{}^{i} - \frac{3}{4}K_{aij}K_{bkl}K^{bij}K^{akl} ,\\
\Delta_2^{(3)} =& + 3R^{abRS}R_{abRS} && \text{(C.4b)}\\
& - 6K_{ai}{}^{k}K_{bjk}\left(R^{aibj} - R^{biaj}\right) - 6K_{aik}K^{ajk}R^{bi}{}_{bj} + 3K_{ai}{}^{j}K_{bj}{}^{k}K^{a}{}_{k}{}^{l}K^{b}{}_{l}{}^{i}\\
& - 6K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K^{b}{}_{l}{}^{i} ,\\
\Delta_3^{(3)} =& + \frac{1}{2}R^{aMNR}R_{aMNR} + 2R^{a\lambda}R^{b}{}_{ab\lambda} && \text{(C.4c)}
\end{aligned}$$

$$
\begin{aligned}
& \quad - K^{a}{}_{i}{}^{k} K_{ajk} R^{ij} - \frac{1}{2} K^{aij} K_{aij} R^{b}{}_{b} \;,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_4^{(3)} &= + R_{MNRS} R^{MNRS} + 2 R R^{ab}{}_{ab} \\
&\quad - 2 K^{aij} K_{aij} R \;,
\end{aligned} \tag{C.4d}
$$

$$
\Delta_5^{(3)} = + R_{M}{}^{N} R^{aM}{}_{aN} - \frac{1}{2} R^{ab} R_{ab} + \frac{1}{2} R^{a}{}_{a} R^{b}{}_{b} \;, \tag{C.4e}
$$

$$
\Delta_6^{(3)} = + \frac{3}{2} R^{aM} R_{aM} \;, \tag{C.4f}
$$

$$
\Delta_7^{(3)} = + R_{MN} R^{MN} + R^{a}{}_{a} R \;, \tag{C.4g}
$$

$$
\Delta_8^{(3)} = + 3R^2 \;. \tag{C.4h}
$$

In each case, the first line corresponds to the Wald-like piece whereas the rest come from the anomaly one, and we have already made use of the RT on-shell condition, $K^a = 0$.

The action for the general quartic theory is:

$$
I_E^{\mathrm{Riem}^4} = -\frac{1}{16\pi G_N} \int \mathrm{d}^{d+1}x \sqrt{G} \left[\frac{d(d-1)}{L^2} + R + L^6 \sum_{i=1}^{26} \gamma_i \mathcal{L}_i^{(4)} \right] \;, \tag{C.5}
$$

with

$$
\mathcal{L}_{26}^{(4)} \equiv R^4 \;, \qquad \mathcal{L}_{25}^{(4)} \equiv R^2 R_{MN} R^{MN} , \tag{C.6a}
$$

$$
\mathcal{L}_{24}^{(4)} \equiv R R_{M}{}^{N} R_{N}{}^{R} R_{R}{}^{M} \;, \qquad \mathcal{L}_{23}^{(4)} \equiv R_{MN} R^{MN} R_{RS} R^{RS} \;, \tag{C.6b}
$$

$$
\mathcal{L}_{22}^{(4)} \equiv R_{M}{}^{N} R_{N}{}^{R} R_{R}{}^{S} R_{S}{}^{M} \;, \qquad \mathcal{L}_{21}^{(4)} \equiv R R_{MNRS} R^{MR} R^{NS} \;, \tag{C.6c}
$$

$$
\mathcal{L}_{20}^{(4)} \equiv R^{MN} R_{MRNS} R^{PR} R_{P}{}^{S} \;, \qquad \mathcal{L}_{19}^{(4)} \equiv R^2 R_{MNRS} R^{MNRS} \;, \tag{C.6d}
$$

$$
\mathcal{L}_{18}^{(4)} \equiv R R_{MNRS} R^{MNR}{}_{P} R^{SP} \;, \qquad \mathcal{L}_{17}^{(4)} \equiv R_{PQ} R^{PQ} R_{MNRS} R^{MNRS} \;, \tag{C.6e}
$$

$$
\mathcal{L}_{16}^{(4)} \equiv R^{MN} R_{N}{}^{R} R^{SPQ}{}_{M} R_{SPQR} \;, \qquad \mathcal{L}_{15}^{(4)} \equiv R^{MN} R^{RS} R^{PQ}{}_{MR} R_{PQNS} \;, \tag{C.6f}
$$

$$
\mathcal{L}_{14}^{(4)} \equiv R^{MN} R^{RS} R^{P}{}_{M}{}^{Q}{}_{N} R_{PRQS} \;, \qquad \mathcal{L}_{13}^{(4)} \equiv R^{MN} R^{RS} R^{P}{}_{M}{}^{Q}{}_{R} R_{PNQS} \;, \tag{C.6g}
$$

$$
\mathcal{L}_{12}^{(4)} \equiv R R_{MN}{}^{RS} R_{RS}{}^{PQ} R_{PQ}{}^{MN} \;, \qquad \mathcal{L}_{11}^{(4)} \equiv R R_{M}{}^{R}{}_{N}{}^{S} R_{R}{}^{P}{}_{S}{}^{Q} R_{P}{}^{M}{}_{Q}{}^{N} \;, \tag{C.6h}
$$

$$
\mathcal{L}_{10}^{(4)} \equiv R^{MN} R_{M}{}^{R}{}_{N}{}^{S} R_{PQUR} R^{PQU}{}_{S} \;, \qquad \mathcal{L}_{9}^{(4)} \equiv R^{MN} R^{RSPQ} R_{RS}{}^{U}{}_{M} R_{PQUN} \;, \tag{C.6i}
$$

$$
\mathcal{L}_{8}^{(4)} \equiv R^{MN} R^{RSPQ} R_{R}{}^{U}{}_{PM} R_{SUQN} \;, \qquad \mathcal{L}_{7}^{(4)} \equiv R_{MNRS} R^{MNRS} R_{PQUT} R^{PQUT} \;, \tag{C.6j}
$$

$$
\mathcal{L}_{6}^{(4)} \equiv R^{MNRS} R_{MNR}{}^{P} R_{QUTS} R^{QUT}{}_{P} \;, \qquad \mathcal{L}_{5}^{(4)} \equiv R^{MNRS} R_{MN}{}^{PQ} R_{PQ}{}^{TU} R_{RSTU} \;, \tag{C.6k}
$$

$$
\mathcal{L}_{4}^{(4)} \equiv R^{MNRS} R_{MN}{}^{PQ} R_{RP}{}^{TU} R_{SQTU} \;, \qquad \mathcal{L}_{3}^{(4)} \equiv R^{MNRS} R_{MN}{}^{PQ} R_{R}{}^{T}{}_{P}{}^{U} R_{STQU} \;, \tag{C.6l}
$$

$$
\mathcal{L}_{2}^{(4)} \equiv R^{MNRS} R_{M}{}^{P}{}_{R}{}^{Q} R_{P}{}^{T}{}_{Q}{}^{U} R_{NTSU} \;, \qquad \mathcal{L}_{1}^{(4)} \equiv R^{MNRS} R_{M}{}^{P}{}_{R}{}^{Q} R_{P}{}^{T}{}_{N}{}^{U} R_{QTSU} \;. \tag{C.6m}
$$

The entanglement entropy functional has the form:

$$
S_{\mathrm{EE}}^{\mathrm{Riem}^4} = \frac{\mathrm{Area}(\Gamma_A)}{4G_N} + \frac{L^6}{4G_N} \int_{\Gamma_A} \mathrm{d}^{d-1}y \sqrt{h} \sum_{i=1}^{26} \gamma_i \Delta_i^{(4)} + \mathcal{O}(\gamma_i^2) \,, \tag{C.7}
$$

with

$$\Delta_{26}^{(4)} = +\,4R^3 \ , \tag{C.8a}$$

$$\Delta_{25}^{(4)} = +\,2RR_{MN}R^{MN} + R^2 R^a{}_a \ , \tag{C.8b}$$

$$\Delta_{24}^{(4)} = +\,R_M{}^N R_N{}^R R_R{}^M + \frac{3}{2} R R^{aM} R_{aM} \ , \tag{C.8c}$$

$$\Delta_{23}^{(4)} = +\,2R^a{}_a R_{MN} R^{MN} \ , \tag{C.8d}$$

$$\Delta_{22}^{(4)} = +\,2R^{aM} R_a{}^N R_{MN} \ , \tag{C.8e}$$

$$\Delta_{21}^{(4)} = +\,R_{MNRS}R^{MR}R^{NS} + RR^{aM}{}_{aN}R^N{}_M - \frac{1}{2}RR^{ab}R_{ab} + \frac{1}{2}RR^a{}_a R^b{}_b \ , \tag{C.8f}$$

$$\Delta_{20}^{(4)} = -\,R_{aMNR}R^{aR}R^{MN} + \frac{1}{2}R^{aM}{}_{aN}R^R_M R^N_R - \frac{1}{2}R^a_M R^b_a R^M_b + \frac{1}{2}R^a_a R^b_M R^M_b \ , \tag{C.8g}$$

$$\begin{aligned}\Delta_{19}^{(4)} = &+\,2RR_{MNRS}R^{MNRS} + 2R^2 R^{ab}{}_{ab} \\ &-\,2R^2 K^{aij}K_{aij} \ ,\end{aligned} \tag{C.8h}$$

$$\begin{aligned}\Delta_{18}^{(4)} = &+\,R_{MNRS}R^{MNR}{}_P R^{SP} + \frac{1}{2}RR_{aMNR}R^{aMNR} + 2RR^{ab}{}_{aM}R^M{}_b \\ &-\,\frac{1}{2}RR^a{}_a K^{bij}K_{bij} - RR^{ij}K_{aik}K^a{}_j{}^k \ ,\end{aligned} \tag{C.8i}$$

$$\begin{aligned}\Delta_{17}^{(4)} = &+\,R^a{}_a R_{MNRS}R^{MNRS} + 2R^{ab}{}_{ab}R_{MN}R^{MN} \\ &+\,2K^{aij}K_{aij}\left(R_{bk}R^{bk} + \frac{2}{3}R_{bc}R^{bc} - R_{MN}R^{MN} - \frac{1}{3}R^b{}_b R^c{}_c\right) \ ,\end{aligned} \tag{C.8j}$$

$$\begin{aligned}\Delta_{16}^{(4)} = &+\,R_M{}^a R_{aNRS}R^{MNRS} + 2R_M{}^a R^b{}_{abN}R^{MN} \\ &-\,\frac{1}{2}K^{aij}K_{aij}\left(\frac{1}{2}R_{bk}R^{bk} + \frac{1}{3}R_{bc}R^{bc} + \frac{1}{3}R^b{}_b R^c{}_c\right) - K^a{}_i{}^k K_{ajk}\left(R^{il}R^j{}_l + \frac{1}{2}R^{ib}R^j{}_b\right) \ ,\end{aligned} \tag{C.8k}$$

$$\begin{aligned}\Delta_{15}^{(4)} = &+\,R_{aMRS}R^a{}_N{}^{RS}R^{MN} + 2R^{aM}R^{bN}R_{abMN} \\ &-\,K^a{}_i{}^k K_{ajk}\left(R^b{}_b R^{ij} - \frac{1}{2}R^{ib}R^j{}_b\right) - K^a{}_i{}^k K^b{}_{jk}R^{[i}{}_a R^{j]}{}_b \ ,\end{aligned} \tag{C.8l}$$

$$\begin{aligned}\Delta_{14}^{(4)} = &+\,R^a{}_{MaR}R_{NS}R^{MNRS} + R^a{}_a R^b{}_{MbN}R^{MN} - R_{aMbN}R^{ab}R^{MN} \\ &+\,\frac{1}{24}K^{aij}K_{aij}\left(R^b{}_b R^c{}_c - 2R_{bc}R^{bc}\right) - \frac{1}{4}K^a{}_i{}^k K_{ajk}R^{ib}R^j{}_b - \frac{1}{2}K^a{}_i{}^k K^b{}_{jk}R^{[i}{}_a R^{j]}{}_b \\ &-\,\frac{1}{2}K^a{}_{ij}K_{akl}R^{ij}R^{kl} \ ,\end{aligned} \tag{C.8m}$$

$$\begin{aligned}\Delta_{13}^{(4)} = &+\,R^a{}_{MRN}R_a{}^M{}_S{}^N R^{RS} - R_{aMbN}R^{aN}R^{bM} + R^a{}_{MaN}R^{Mb}R^N{}_b \\ &-\,\frac{1}{8}K^{aij}K_{aij}R^b{}_b R^c{}_c - \frac{1}{2}K^a{}_i{}^k K_{ajk}R^b{}_b R^{ij} - \frac{1}{2}K^a{}_{ij}K_{akl}R^{ik}R^{jl} \ ,\end{aligned} \tag{C.8n}$$

$$\begin{aligned}\Delta_{12}^{(4)} = &+\,R_{MN}{}^{RS}R_{RS}{}^{PQ}R_{PQ}{}^{MN} + 3RR_{abMN}R^{abMN} \\ &-\,6K_{ai}{}^k K_{bjk}\left(R^{aibj} - R^{biaj}\right)R - 6K^a{}_i{}^k K_{ajk}R^{bi}{}_b{}^j R + 3K_{ai}{}^j K_{bj}{}^k K^a{}_k{}^l K^b{}_l{}^i R \\ &-\,6K_{ai}{}^j K^a{}_j{}^k K_{bk}{}^l K^b{}_l{}^i R \ ,\end{aligned} \tag{C.8o}$$

$$\begin{aligned}\Delta_{11}^{(4)} = &+\,R_M{}^R{}_N{}^S R_R{}^P{}_S{}^Q R_P{}^M{}_Q{}^N + \frac{3}{2}RR^{aM}{}_{aN}R^{bN}{}_{bM} - \frac{3}{2}RR^{aMbN}R_{bMaN} \\ &-\,\frac{3}{2}K^a{}_{ij}K_{akl}R^{ikjl}R - 3K_{ai}{}^k K_{bjk}R^{abij}R + \frac{3}{4}K^{aij}K_{aij}R^{bc}{}_{bc}R - \frac{3}{8}K^{aij}K_{aij}K^{bkl}K_{bkl}R\end{aligned} \tag{C.8p}$$

$$
\begin{aligned}
&- \frac{3}{4} K_{aij} K_{bkl} K^{bij} K^{akl} R + \frac{9}{4} K_{ai}{}^{j} K_{bj}{}^{k} K^{a}{}_{k}{}^{l} K^{b}{}_{l}{}^{i} R - \frac{3}{2} K_{ai}{}^{j} K^{a}{}_{j}{}^{k} K_{bk}{}^{l} K^{b}{}_{l}{}^{i} R \,, \\
\Delta_{10}^{(4)} =& + \frac{1}{2} R^{a}{}_{MaN} R^{M}{}_{RSP} R^{NRSP} + 2 R^{M}_{N} R_{M}{}^{aNR} R^{b}{}_{abR} + \frac{1}{2} R^{a}_{a} R_{bMNR} R^{bMNR} - \frac{1}{2} R^{a}_{b} R_{aMNR} R^{bMNR} \\
&+ K_{aij} K_{bkl} R^{i[a} R^{b]kjl} - K_{aij} K^{a}{}_{kl} \left(R^{ij} R_{b}{}^{kbl} - \frac{1}{2} R^{ib} R_{b}{}^{kjl} \right) \qquad \text{(C.8q)} \\
&- K_{ai}{}^{k} K_{bjk} R^{i[a} R_{c}{}^{b]cj} - \frac{1}{2} K^{aij} K_{aij} \left(R^{kl} R^{b}{}_{kbl} + R^{bk} R^{c}{}_{bck} + \frac{1}{2} R^{b}{}_{b} R^{cd}{}_{cd} \right) \\
&+ K_{ai}{}^{k} K^{a}{}_{jk} \left(R_{bl} R^{bijl} - \frac{1}{2} R_{b}{}^{i} R_{c}{}^{bcj} + \frac{2}{3} R_{bc} R^{bicj} - \frac{1}{6} R_{b}{}^{b} R_{c}{}^{icj} - R_{lm} R^{iljm} \right) \\
&+ \frac{1}{2} K_{al}{}^{i} K_{bi}{}^{j} K^{b}{}_{j}{}^{k} K^{a}{}_{km} R^{lm} - K_{ai}{}^{j} K^{a}{}_{j}{}^{k} K_{bk}{}^{i} K^{b}{}_{lm} R^{lm} - \frac{1}{4} K^{aij} K_{aij} K_{bl}{}^{k} K^{b}{}_{mk} R^{lm} \\
&+ \frac{1}{8} K^{aij} K_{aij} K^{bkl} K_{bkl} R^{c}{}_{c} - \frac{1}{4} K_{ai}{}^{j} K^{a}{}_{j}{}^{k} K_{bk}{}^{l} K^{b}{}_{l}{}^{i} R^{c}{}_{c} \,, \\
\Delta_{9}^{(4)} =& + \frac{1}{2} R^{MNRS} R_{MN}{}^{Ta} R_{RSTa} + R^{MN} R^{abR}{}_{M} R_{abRN} - 2 R^{Ma} R^{NRb}{}_{M} R_{NRab} \qquad \text{(C.8r)} \\
&+ 2 K_{aij} K_{bkl} R^{ik} R^{j[ab]l} - K_{aij} K^{a}{}_{kl} R^{ik} R_{b}{}^{jbl} + K_{ai}{}^{k} K_{bjk} \Big(-4 R^{il} R^{j[ab]}{}_{l} + R^{ic} R^{abj}{}_{c} \\
&-2 R^{al} R^{b[ij]}{}_{l} + 6 R^{c[a} R^{b]ji}{}_{c} \Big) + K_{ai}{}^{k} K^{a}{}_{jk} \Big(-2 R^{il} R^{bj}{}_{bl} - R_{b}{}^{i} R_{c}{}^{bcj} + R_{bl} R^{bijl} \\
&- \frac{2}{3} R_{bc} R^{bicj} - \frac{7}{6} R_{b}{}^{b} R_{c}{}^{icj} \Big) - \frac{3}{2} K_{al}{}^{i} K_{bi}{}^{j} K^{b}{}_{j}{}^{k} K^{a}{}_{km} R^{lm} + \frac{3}{2} K_{al}{}^{i} K_{bi}{}^{j} K^{a}{}_{j}{}^{k} K^{b}{}_{km} R^{lm} \\
&- \frac{3}{2} K_{al}{}^{i} K^{a}{}_{i}{}^{j} K_{bj}{}^{k} K^{b}{}_{km} R^{lm} + \frac{3}{4} K_{ai}{}^{j} K_{bj}{}^{k} K^{a}{}_{k}{}^{l} K^{b}{}_{l}{}^{i} R^{c}{}_{c} - \frac{3}{2} K_{ai}{}^{j} K^{a}{}_{j}{}^{k} K_{bk}{}^{l} K^{b}{}_{l}{}^{i} R^{c}{}_{c} \,, \\
\Delta_{8}^{(4)} =& + \frac{1}{2} R^{MNRS} R_{M}{}^{a}{}_{R}{}^{T} R_{NaST} + R^{MN} R^{aR}{}_{[a|M} R^{b}{}_{R|b]N} + 2 R^{aM} R_{M}{}^{N}{}_{[a|}{}^{R} R^{b}{}_{N|b]R} \qquad \text{(C.8s)} \\
&- \frac{1}{2} K_{aij} K_{bkl} \left(R^{i[a} R^{b]kjl} + R^{ik} R^{abjl} \right) + K_{aij} K^{a}{}_{kl} \left(R^{im} R^{jkl}{}_{m} - \frac{1}{4} R_{b}{}^{i} R^{bkjl} - \frac{1}{4} R_{b}{}^{b} R^{ikjl} \right) \\
&- K_{ai}{}^{k} K_{bjk} \left(R^{jl} R^{abi}{}_{l} - \frac{1}{2} R^{al} R^{ijb}{}_{l} + \frac{1}{2} R^{i[a} R_{c}{}^{b]cj} + \frac{3}{4} R_{c}{}^{c} R^{abij} \right) \\
&+ \frac{1}{4} K_{ai}{}^{k} K^{a}{}_{jk} \left(R^{ij} R^{bc}{}_{bc} - R_{b}{}^{i} R_{c}{}^{bcj} \right) + \frac{1}{4} K^{aij} K_{aij} \left(R^{bk} R^{c}{}_{bck} + R^{b}{}_{b} R^{cd}{}_{cd} \right) \\
&- \frac{1}{2} K_{al}{}^{i} K_{bi}{}^{j} K^{b}{}_{j}{}^{k} K^{a}{}_{km} R^{lm} + \frac{5}{4} K_{al}{}^{i} K_{bi}{}^{j} K^{a}{}_{j}{}^{k} K^{b}{}_{km} R^{lm} - \frac{1}{4} K_{al}{}^{i} K^{a}{}_{i}{}^{j} K_{bj}{}^{k} K^{b}{}_{km} R^{lm} \\
&- \frac{1}{2} K_{a}{}^{ij} K_{bij} K^{a}{}_{l}{}^{k} K^{b}{}_{mk} R^{lm} - \frac{1}{8} K^{aij} K_{aij} K_{bl}{}^{k} K^{b}{}_{mk} R^{lm} - \frac{1}{8} K_{aij} K_{bkl} K^{bij} K^{akl} R^{c}{}_{c} \\
&- \frac{1}{8} K^{aij} K_{aij} K^{bkl} K_{bkl} R^{c}{}_{c} + \frac{1}{2} K_{ai}{}^{j} K_{bj}{}^{k} K^{a}{}_{k}{}^{l} K^{b}{}_{l}{}^{i} R^{c}{}_{c} - \frac{3}{8} K_{ai}{}^{j} K^{a}{}_{j}{}^{k} K_{bk}{}^{l} K^{b}{}_{l}{}^{i} R^{c}{}_{c} \,, \\
\Delta_{7}^{(4)} =& + 4 R_{MNRS} R^{MNRS} R^{ab}{}_{ab} \qquad \text{(C.8t)} \\
&+ \frac{64}{3} K_{aij} K_{bkl} R^{iaj[b|} R^{k|c]l}{}_{c} - \frac{8}{3} K_{aij} K^{a}{}_{kl} R^{ibj}{}_{b} R^{kcl}{}_{c} - 4 K^{aij} K_{aij} \Big(R^{bc}{}_{bc} R^{de}{}_{de} + 2 R^{bcdk} R_{bcdk} \\
&+ 2 R^{bckl} R_{bckl} + \frac{8}{3} R^{bkcl} R_{bkcl} - \frac{4}{3} R^{bkcl} R_{ckbl} + \frac{4}{3} R_{b}{}^{kbl} R^{c}{}_{kcl} + 2 R^{bklm} R_{bklm} + R^{klmn} R_{klmn} \Big) \\
&- 8 K^{aij} K_{aij} K_{bkl} K^{b}{}_{mn} R^{kmln} - 16 K^{aij} K_{aij} K_{bk}{}^{m} K_{clm} \left(R^{[b|k|c]l} + R^{bckl} \right) \\
&- 8 K^{aij} K_{aij} K_{bk}{}^{m} K^{b}{}_{lm} R_{c}{}^{kcl} + 4 K^{aij} K_{aij} K^{bkl} K_{bkl} R^{cd}{}_{cd}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{32}{3}K^{aij}K_{aij}K_{bk}{}^{l}K_{cl}{}^{m}K^{b}{}_{m}{}^{n}K^{c}{}_{n}{}^{k}-\frac{32}{3}K^{aij}K_{aij}K_{bk}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{k}\\
&-\frac{8}{3}K^{aij}K_{aij}K_{bkl}K_{cmn}K^{ckl}K^{bmn}-\frac{4}{3}K^{aij}K_{aij}K^{bkl}K_{bkl}K^{cmn}K_{cmn}\ ,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_6^{(4)}=&+4R^{MNRS}R^{a}{}_{NRS}R^{b}{}_{abM}\\
&+2K_{aij}K_{bkl}\left(R^{[b|ijm}R^{|a]kl}{}_{m}+\frac{4}{3}R^{iaj[b|}R^{k|c]l}{}_{c}+2R^{ikj[a|}R^{lc|b]}{}_{c}\right)\\
&+K_{aij}K^{a}{}_{kl}\left(\frac{1}{3}R^{ibk}{}_{b}R^{jcl}{}_{c}-\frac{2}{3}R^{ibkc}R^{(j}{}_{b}{}^{l)}{}_{c}-\frac{7}{3}R^{ibj}{}_{b}R^{kcl}{}_{c}-R^{ibjm}R^{k}{}_{b}{}^{l}{}_{m}+2R^{ikjb}R^{lc}{}_{bc}\right)\\
&+K_{ai}{}^{k}K_{bjk}\left(R^{i[b|cd}R^{j|a]}{}_{cd}+\frac{4}{3}R^{i[b|cl}R^{j|a]}{}_{cl}+\frac{4}{3}R^{i[ab]l}R^{jc}{}_{cl}\right)\\
&+K_{ai}{}^{k}K^{a}{}_{jk}\left(-\frac{3}{2}R^{ibcd}R^{j}{}_{bcd}+2R^{ib}{}_{l[c}R^{jcl}{}_{b]}-3R^{iblc}R^{j}{}_{blc}-2R^{ilbc}R^{j}{}_{lbc}-R^{iblm}R^{j}{}_{blm}\right.\\
&\left.-2R^{ilbm}R^{j}{}_{lbm}-2R^{ilmn}R^{j}{}_{lmn}\right)+K^{aij}K_{aij}\left(-R^{bc}{}_{bc}R^{de}{}_{de}-\frac{3}{2}R^{bcdk}R_{bcdk}\right.\\
&\left.-R^{bckl}R_{bckl}-\frac{4}{3}R^{bkcl}R_{bkcl}+\frac{2}{3}R^{bkcl}R_{ckbl}-\frac{2}{3}R_{b}{}^{kbl}R^{c}{}_{kcl}-\frac{1}{2}R^{bklm}R_{bklm}\right)\\
&-4K_{ai}{}^{k}K_{bk}{}^{l}K^{b}{}_{lj}K^{a}{}_{mn}R^{imjn}-4K_{ai}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{c}{}_{mj}\left(R^{abij}+R^{[a|i|b]j}\right)\\
&-2K_{ai}{}^{k}K^{a}{}_{k}{}^{l}K_{bl}{}^{m}K^{b}{}_{mj}R_{c}{}^{icj}-2K^{aij}K_{aij}K_{bk}{}^{m}K_{clm}\left(R^{bckl}+R^{[b|k|c]l}\right)\\
&-K^{aij}K_{aij}K_{bk}{}^{m}K^{b}{}_{lm}R_{c}{}^{kcl}-2K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{i}K^{b}{}_{lm}R_{c}{}^{lcm}+K^{aij}K_{aij}K^{bkl}K_{bkl}R^{cd}{}_{cd}\\
&+\frac{10}{3}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{b}{}_{m}{}^{n}K^{c}{}_{n}{}^{i}-2K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{c}{}_{m}{}^{n}K^{b}{}_{n}{}^{i}\\
&-\frac{2}{3}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{i}-\frac{2}{3}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{i}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{l}\\
&-\frac{4}{3}K_{a}{}^{ij}K_{bij}K^{a}{}_{k}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{k}+K^{aij}K_{aij}K_{bk}{}^{l}K_{cl}{}^{m}K^{b}{}_{m}{}^{n}K^{c}{}_{n}{}^{k}\\
&-\frac{4}{3}K^{aij}K_{aij}K_{bk}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{k}-\frac{1}{3}K^{aij}K_{aij}K^{bkl}K_{bkl}K^{cmn}K_{cmn}\ ,
\end{aligned}
\tag{C.8u}
$$

$$
\begin{aligned}
\Delta_5^{(4)}=&+4R^{MNRS}R_{MN}{}^{ab}R_{RSab}\\
&+16K_{aij}K_{bkl}R^{i[ab]k}R^{jcl}{}_{c}+4K_{aij}K^{a}{}_{kl}\left(2R^{ibkc}R^{j}{}_{[b}{}^{l}{}_{c]}-R^{ibk}{}_{b}R^{jcl}{}_{c}\right)\\
&+8K_{ai}{}^{k}K_{bjk}\left(R^{i[b|cd}R^{j|a]}{}_{cd}+\frac{4}{3}R^{i[b|cl}R^{j|a]}{}_{cl}+\frac{8}{3}R^{i[ab]l}R^{jc}{}_{lc}+R^{[b|ilm}R^{|a]j}{}_{lm}\right)\\
&-4K_{ai}{}^{k}K^{a}{}_{jk}\left(R^{ibcd}R^{j}{}_{bcd}+\frac{8}{3}R^{iblc}R^{j}{}_{blc}-\frac{8}{3}R^{ibl}{}_{[c|}R^{jc}{}_{l|b]}+R^{iblm}R^{j}{}_{blm}\right)\\
&+48K_{ai}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{c}{}_{mj}R^{i[ab]j}+24K_{ai}{}^{k}K_{ck}{}^{l}K_{bl}{}^{m}K^{c}{}_{mj}R^{aibj}\\
&-12K_{ai}{}^{k}K_{ck}{}^{l}K^{c}{}_{l}{}^{m}K_{bmj}R^{aibj}-12K_{ci}{}^{k}K_{ak}{}^{l}K_{bl}{}^{m}K^{c}{}_{mj}R^{aibj}\\
&-12K_{ai}{}^{k}K^{a}{}_{k}{}^{l}K_{bl}{}^{m}K^{b}{}_{mj}R_{c}{}^{icj}+12K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{b}{}_{m}{}^{n}K^{c}{}_{n}{}^{i}\\
&-12K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{c}{}_{m}{}^{n}K^{b}{}_{n}{}^{i}-4K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{i}\ ,
\end{aligned}
\tag{C.8v}
$$

$$
\Delta_4^{(4)}=+4R^{abMN}R_{aM}{}^{RS}R_{bNRS}
\tag{C.8w}
$$

$$
\begin{aligned}
&+ 4K_{aij}K_{bkl}\left(2R^{i[ab]k}R^{jcl}{}_{c} - R^{[a|ikm}R^{|b]lj}{}_{m}\right) + 2K_{aij}K^{a}{}_{kl}\Bigg(\frac{4}{3}R^{ibk}{}_{[c}R^{lcj}{}_{b]} \\
&-\frac{4}{3}R^{ibkc}R^{l\ j}_{\ b\ c} - R^{ibkm}R^{l\ j}_{\ b\ m}\Bigg) + 2K_{ai}{}^{k}K_{bjk}\Bigg(-2R^{abij}R^{cd}{}_{cd} + \frac{4}{3}R^{i[a|cl}R^{j|b]}{}_{cl} \\
&+\frac{4}{3}R^{i[ab]l}R^{jc}{}_{lc} - 4R^{ijal}R^{bc}{}_{lc} + R^{[a|ilm}R^{|b]j}{}_{lm} - 2R^{ablm}R^{ij}{}_{lm}\Bigg) \\
&+ K_{ai}{}^{k}K^{a}{}_{jk}\left(-2R^{ibcd}R^{j}{}_{bcd} - \frac{14}{3}R^{iblc}R^{j}{}_{blc} + \frac{20}{3}R^{ibl}{}_{[c|}R^{jc}{}_{l|b]} - R^{iblm}R^{j}{}_{blm}\right) \\
&- 4K_{ai}{}^{m}K_{bjm}K^{a}{}_{k}{}^{n}K^{b}{}_{ln}R^{ijkl} + 4K_{ai}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{c}{}_{mj}\left(R^{abij} + 4R^{i[ab]j}\right) \\
&+ 4K_{ai}{}^{k}K_{ck}{}^{l}K_{bl}{}^{m}K^{c}{}_{mj}\left(R^{aibj} - R^{abij}\right) - 2K_{ai}{}^{k}K_{ck}{}^{l}K^{c}{}_{l}{}^{m}K_{bmj}R^{aibj} \\
&- 2K_{ci}{}^{k}K_{ak}{}^{l}K_{bl}{}^{m}K^{c}{}_{mj}R^{aibj} - 6K_{ai}{}^{k}K^{a}{}_{k}{}^{l}K_{bl}{}^{m}K^{b}{}_{mj}R_{c}{}^{icj} \\
&+ 2K^{clm}K_{clm}K_{ai}{}^{k}K_{bjk}R^{abij} + 2K_{ai}{}^{j}K_{bj}{}^{k}K^{a}{}_{k}{}^{l}K^{b}{}_{l}{}^{i}R^{cd}{}_{cd} \\
&- 2K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K^{b}{}_{l}{}^{i}R^{cd}{}_{cd} + \frac{2}{3}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{b}{}_{m}{}^{n}K^{c}{}_{n}{}^{i} \\
&+ 2K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{c}{}_{m}{}^{n}K^{b}{}_{n}{}^{i} - \frac{14}{3}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{i} \\
&- \frac{4}{3}K^{aij}K_{aij}K_{bk}{}^{l}K_{cl}{}^{m}K^{b}{}_{m}{}^{n}K^{c}{}_{n}{}^{k} + \frac{4}{3}K^{aij}K_{aij}K_{bk}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{k}\,,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_3^{(4)} =& + 2R^{abMN}R_{a}{}^{R}{}_{M}{}^{S}R_{bRNS} + R^{MaRS}R^{N}{}_{aRS}R^{b}{}_{MbN} - R^{MaRS}R^{Nb}{}_{RS}R_{MbNa} \\
&- 2K_{aij}K_{bkl}\left(R^{ikj[a|}R^{lc|b]}{}_{c} + R^{ikab}R^{jcl}{}_{c} + 2R^{i[ab]m}R^{jkl}{}_{m} + R^{[a|ikm}R^{jl|b]}{}_{m}\right) \\
&+ K_{aij}K^{a}{}_{kl}\Bigg(2R^{ikbc}R^{j\ l}_{\ b\ c} - \frac{1}{2}R^{ikbc}R^{jl}{}_{bc} - R^{ikjb}R^{lc}{}_{bc} + R^{ikbm}R^{j\ l}_{\ b\ m} - \frac{1}{2}R^{ikbm}R^{jl}{}_{bm} \\
&+2R^{ibm}{}_{b}R^{jkl}{}_{m} - \frac{1}{2}R^{ikmn}R^{jl}{}_{mn}\Bigg) - K_{ai}{}^{k}K_{bjk}\Bigg(R^{i[ab]j}R^{cd}{}_{cd} + R^{abij}R^{cd}{}_{cd} \\
&+\frac{1}{2}R^{i[a|cd}R^{j|b]}{}_{cd} + 2R^{ijal}R^{bc}{}_{lc} + R^{[a|lim}R^{|b]}{}_{m}{}^{j}{}_{l} + R^{ablm}R^{ij}{}_{lm} + 2R^{albm}R^{i\ j}_{\ [l\ m]}\Bigg) \\
&+ K_{ai}{}^{k}K^{a}{}_{jk}\left(\frac{1}{2}R_{b}{}^{ibj}R^{cd}{}_{cd} - \frac{1}{4}R^{ibcd}R^{j}{}_{bcd} - R^{b}{}_{lbm}R^{iljm} + \frac{1}{2}R^{ilbm}R^{j}{}_{mbl}\right) \\
&+ \frac{1}{4}K^{aij}K_{aij}\left(R^{bc}{}_{bc}R^{de}{}_{de} + R^{bcdk}R_{bcdk} + R^{bckl}R_{bckl}\right) \\
&+ K_{ai}{}^{m}K_{bjm}K^{a}{}_{k}{}^{n}K^{b}{}_{ln}\left(\frac{1}{2}R^{iljk} - R^{ijkl} - \frac{3}{2}R^{ikjl}\right) - \frac{1}{2}K_{ai}{}^{m}K^{a}{}_{jm}K_{bk}{}^{n}K^{b}{}_{ln}R^{ikjl} \\
&+ K_{aij}K_{bk}{}^{m}K^{a}{}_{m}{}^{n}K^{b}{}_{nl}R^{ikjl} - 2K_{aij}K^{a}{}_{k}{}^{m}K_{bm}{}^{n}K^{b}{}_{nl}R^{ikjl} \\
&- K_{ai}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{c}{}_{mj}\left(2R^{biaj} + 3R^{aibj}\right) + K_{ai}{}^{k}K_{ck}{}^{l}K_{bl}{}^{m}K^{c}{}_{mj}\left(3R^{aibj} + R^{biaj} - 2R^{abij}\right) \\
&+ \frac{1}{2}K_{ai}{}^{k}K_{ck}{}^{l}K^{c}{}_{l}{}^{m}K_{bmj}R^{aibj} + \frac{1}{2}K_{ci}{}^{k}K_{ak}{}^{l}K_{bl}{}^{m}K^{c}{}_{mj}R^{aibj} + K_{a}{}^{lm}K_{blm}K_{ci}{}^{k}K^{c}{}_{jk}R^{aibj} \\
&- 2K^{clm}K_{blm}K_{ai}{}^{k}K_{cjk}R^{aibj} + K^{clm}K_{clm}K_{ai}{}^{k}K_{bjk}\left(R^{abij} + \frac{3}{4}R^{biaj} + \frac{1}{4}R^{aibj}\right) \\
&+ \frac{3}{2}K_{ai}{}^{k}K^{a}{}_{k}{}^{l}K_{bl}{}^{m}K^{b}{}_{mj}R_{c}{}^{icj} - \frac{1}{2}K_{ak}{}^{l}K_{bl}{}^{m}K^{b}{}_{m}{}^{k}K^{a}{}_{ij}R_{c}{}^{icj} - \frac{5}{4}K^{blm}K_{blm}K_{ai}{}^{k}K^{a}{}_{jk}R_{c}{}^{icj} \\
&+ \frac{1}{4}K_{ai}{}^{j}K_{bj}{}^{k}K^{a}{}_{k}{}^{l}K^{b}{}_{l}{}^{i}R^{cd}{}_{cd} - \frac{1}{4}K^{aij}K_{aij}K^{bkl}K_{bkl}R^{cd}{}_{cd}
\end{aligned}
\tag{C.8x}
$$

$$
\begin{aligned}
&+\frac{2}{3}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{b}{}_{m}{}^{n}K^{c}{}_{n}{}^{i}+2K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{c}{}_{m}{}^{n}K^{b}{}_{n}{}^{i}\\
&-\frac{4}{3}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{i}-\frac{1}{3}K_{ai}{}^{j}K_{bj}{}^{k}K_{ck}{}^{i}K^{a}{}_{l}{}^{m}K^{b}{}_{m}{}^{n}K^{c}{}_{n}{}^{l}\\
&-\frac{1}{3}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{i}K_{cl}{}^{m}K^{c}{}_{m}{}^{n}K^{b}{}_{n}{}^{l}-\frac{2}{3}K_{a}{}^{ij}K_{bij}K^{a}{}_{k}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{k}\\
&-\frac{1}{6}K^{aij}K_{aij}K_{bk}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{k}+\frac{1}{12}K^{aij}K_{aij}K^{bkl}K_{bkl}K^{cmn}K_{cmn}\ ,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_2^{(4)}=&+4R^{MNRS}R^{[a}{}_{MaR}R^{b]}{}_{NbS} \qquad \text{(C.8y)}\\
&+2K_{aij}K_{bkl}\left(\frac{2}{3}R^{iaj[c|}R^{k|b]l}{}_{c}+2R^{ikj[a|}R^{lc|b]}{}_{c}-R^{i[ab]k}R^{jcl}{}_{c}\right)\\
&+K_{aij}K^{a}{}_{kl}\Big(R^{ikjl}R^{bc}{}_{bc}-2R^{ikjb}R^{lc}{}_{bc}-\frac{1}{6}R^{ibj}{}_{b}R^{kcl}{}_{c}-\frac{1}{2}R^{ibk}{}_{b}R^{jcl}{}_{c}-\frac{4}{3}R^{ibjc}R^{k}{}_{b}{}^{l}{}_{c}\\
&+R^{ibkc}R^{[j}{}_{b}{}^{l]}{}_{c}+R^{ikbc}R^{jl}{}_{bc}-2R^{ibjm}R^{k}{}_{b}{}^{l}{}_{m}-2R^{imjn}R^{k}{}_{m}{}^{l}{}_{n}\Big)\\
&-2K_{ai}{}^{k}K_{bjk}\left(\frac{8}{3}R^{i[ab]l}R^{jc}{}_{lc}+\frac{2}{3}R^{i[a|cl}R^{j|b]}{}_{cl}+R^{[a|lim}R^{|b]}{}_{l}{}^{j}{}_{m}\right)\\
&-K_{ai}{}^{k}K^{a}{}_{jk}\left(R^{ibcd}R^{j}{}_{bcd}+\frac{4}{3}R^{iblc}R^{j}{}_{blc}+\frac{4}{3}R^{ibl}{}_{[b|}R^{jc}{}_{l|c]}+2R^{ilbc}R^{j}{}_{lbc}+R^{ilbm}R^{j}{}_{lbm}\right)\\
&-\frac{1}{2}K^{aij}K_{aij}\left(R^{bc}{}_{bc}R^{de}{}_{de}+R^{bcdk}R_{bcdk}+\frac{2}{3}R^{bkcl}R_{(b|k|c)l}-\frac{1}{3}R_{b}{}^{kbl}R^{c}{}_{kcl}\right)\\
&+2K_{aij}K_{bk}{}^{m}K^{a}{}_{m}{}^{n}K^{b}{}_{nl}R^{ikjl}-2K_{a}{}^{mn}K_{bmn}K^{a}{}_{ij}K^{b}{}_{kl}R^{ikjl}-\frac{1}{2}K^{amn}K_{amn}K_{bij}K^{b}{}_{kl}R^{ikjl}\\
&-2K_{ai}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{c}{}_{mj}\left(3R^{abij}+R^{i(ab)j}\right)+K_{ai}{}^{k}K_{ck}{}^{l}K_{bl}{}^{m}K^{c}{}_{mj}\left(2R^{abij}-R^{aibj}-2R^{biaj}\right)\\
&+\frac{1}{2}K_{ai}{}^{k}K_{ck}{}^{l}K^{c}{}_{l}{}^{m}K_{bmj}R^{aibj}+\frac{1}{2}K_{ci}{}^{k}K_{ak}{}^{l}K_{bl}{}^{m}K^{c}{}_{mj}R^{aibj}-\frac{3}{2}K_{ai}{}^{k}K^{a}{}_{k}{}^{l}K_{bl}{}^{m}K^{b}{}_{mj}R_{c}{}^{icj}\\
&-K_{ak}{}^{l}K_{bl}{}^{m}K^{b}{}_{m}{}^{k}K^{a}{}_{ij}R_{c}{}^{icj}-\frac{1}{2}K_{ai}{}^{j}K_{bj}{}^{k}K^{a}{}_{k}{}^{l}K^{b}{}_{l}{}^{i}R^{cd}{}_{cd}+\frac{1}{2}K_{a}{}^{ij}K_{bij}K^{akl}K^{b}{}_{kl}R^{cd}{}_{cd}\\
&+\frac{1}{2}K^{aij}K_{aij}K^{bkl}K_{bkl}R^{cd}{}_{cd}+\frac{5}{6}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{b}{}_{m}{}^{n}K^{c}{}_{n}{}^{i}\\
&-\frac{7}{2}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{c}{}_{m}{}^{n}K^{b}{}_{n}{}^{i}+\frac{3}{2}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{i}\\
&-\frac{1}{3}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{i}K_{cl}{}^{m}K^{c}{}_{m}{}^{n}K^{b}{}_{n}{}^{l}+\frac{4}{3}K_{a}{}^{ij}K_{bij}K^{a}{}_{k}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{k}\\
&+K^{aij}K_{aij}K_{bk}{}^{l}K_{cl}{}^{m}K^{b}{}_{m}{}^{n}K^{c}{}_{n}{}^{k}-\frac{2}{3}K^{aij}K_{aij}K_{bk}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{k}\\
&-\frac{4}{3}K^{aij}K_{aij}K_{b}{}^{kl}K_{ckl}K^{bmn}K^{c}{}_{mn}+\frac{1}{6}K^{aij}K_{aij}K^{bkl}K_{bkl}K^{cmn}K_{cmn}\ ,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_1^{(4)}=&+4R^{aMRN}R^{S}{}_{N[a|M}R^{b}{}_{R|b]S} \qquad \text{(C.8z)}\\
&+K_{aij}K_{bkl}\left(\frac{4}{3}R^{iaj[c|}R^{k|b]l}{}_{c}+2R^{ikj[a|}R^{lc|b]}{}_{c}+4R^{i[ab]m}R^{jkl}{}_{m}+R^{[a|ijm}R^{|b]kl}{}_{m}\right)\\
&+K_{aij}K^{a}{}_{kl}\Big(-R^{ikjb}R^{lc}{}_{bc}-\frac{1}{3}R^{ibk}{}_{b}R^{jcl}{}_{c}-\frac{2}{3}R^{ibjc}R^{k}{}_{b}{}^{l}{}_{c}-\frac{2}{3}R^{ibkc}R^{j}{}_{b}{}^{l}{}_{c}+\frac{1}{3}R^{ibkc}R^{l}{}_{b}{}^{j}{}_{c}
\end{aligned}
$$

$$
\begin{aligned}
&+R^{ikbc}R^{jl}{}_{bc}+2R_{b}{}^{ibm}R^{jkl}{}_{m}-\frac{1}{2}R^{ibjm}R^{k}{}_{b}{}^{l}{}_{m}-R^{ibkm}R^{j}{}_{b}{}^{l}{}_{m}+\frac{1}{2}R^{ikbm}R^{jl}{}_{bm}-R^{imkn}R^{j}{}_{m}{}^{l}{}_{n}\Big)\\
&+K_{ai}{}^{k}K_{bjk}\Big(3R^{i[ab]j}R^{cd}{}_{cd}+\frac{1}{2}R^{i[a|cd}R^{j|b]}{}_{cd}+\frac{2}{3}R^{i[ab]l}R^{jc}{}_{lc}-4R^{i[a|jl}R^{|b]c}{}_{lc}+\frac{2}{3}R^{i[a|cl}R^{j|b]}{}_{cl}\\
&-2R^{albm}R^{i}{}_{[l}{}^{j}{}_{m]}\Big)+K_{ai}{}^{k}K^{a}{}_{jk}\Big(\frac{1}{2}R_{b}{}^{ibj}R^{cd}{}_{cd}-\frac{3}{4}R^{ibcd}R^{j}{}_{bcd}+\frac{1}{6}R^{ibl}{}_{b}R^{jc}{}_{lc}+\frac{1}{3}R^{iblc}R^{j}{}_{(bc)l}\\
&-R^{ilbc}R^{j}{}_{lbc}-R^{iljm}R^{b}{}_{lbm}\Big)+\frac{1}{4}K^{aij}K_{aij}\left(2R^{bkcl}R_{[b|k|c]l}-R_{b}{}^{kbl}R^{c}{}_{kcl}\right)\\
&+K_{ai}{}^{m}K_{bjm}K^{a}{}_{k}{}^{n}K^{b}{}_{ln}R^{i(kl)j}-\frac{1}{2}K_{ai}{}^{m}K^{a}{}_{jm}K_{bk}{}^{n}K^{b}{}_{ln}R^{ikjl}-2K_{aij}K_{bk}{}^{m}K^{a}{}_{m}{}^{n}K^{b}{}_{nl}R^{ikjl}\\
&-2K_{ai}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{c}{}_{mj}\left(2R^{abij}+R^{biaj}\right)+K_{ai}{}^{k}K_{ck}{}^{l}K_{bl}{}^{m}K^{c}{}_{mj}\left(2R^{abij}+3R^{biaj}-R^{aibj}\right)\\
&+K_{a}{}^{lm}K_{blm}K_{ci}{}^{k}K^{c}{}_{jk}R^{aibj}-2K_{a}{}^{lm}K_{clm}K_{bi}{}^{k}K^{c}{}_{jk}R^{biaj}\\
&+\frac{1}{2}K^{clm}K_{clm}K_{ai}{}^{k}K_{bjk}\left(5R^{aibj}-3R^{biaj}\right)+K_{ai}{}^{k}K^{a}{}_{k}{}^{l}K_{bl}{}^{m}K^{b}{}_{mj}R_{c}{}^{icj}\\
&-K_{ak}{}^{l}K_{bl}{}^{m}K^{b}{}_{m}{}^{k}K^{a}{}_{ij}R_{c}{}^{icj}-\frac{3}{2}K^{blm}K_{blm}K_{ai}{}^{k}K^{a}{}_{jk}R_{c}{}^{icj}+\frac{3}{4}K_{ai}{}^{j}K_{bj}{}^{k}K^{a}{}_{k}{}^{l}K^{b}{}_{l}{}^{i}R^{cd}{}_{cd}\\
&-\frac{1}{2}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K^{b}{}_{l}{}^{i}R^{cd}{}_{cd}+\frac{13}{6}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{b}{}_{m}{}^{n}K^{c}{}_{n}{}^{i}\\
&+\frac{1}{2}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K_{cl}{}^{m}K^{c}{}_{m}{}^{n}K^{b}{}_{n}{}^{i}-\frac{7}{6}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{i}\\
&-\frac{1}{3}K_{ai}{}^{j}K_{bj}{}^{k}K_{ck}{}^{i}K^{a}{}_{l}{}^{m}K^{b}{}_{m}{}^{n}K^{c}{}_{n}{}^{l}-\frac{1}{2}K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{i}K_{cl}{}^{m}K^{c}{}_{m}{}^{n}K^{b}{}_{n}{}^{l}\\
&-K_{a}{}^{ij}K_{bij}K^{a}{}_{k}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{k}-\frac{13}{12}K^{aij}K_{aij}K_{bk}{}^{l}K_{cl}{}^{m}K^{b}{}_{m}{}^{n}K^{c}{}_{n}{}^{k}\\
&+\frac{5}{6}K^{aij}K_{aij}K_{bk}{}^{l}K^{b}{}_{l}{}^{m}K_{cm}{}^{n}K^{c}{}_{n}{}^{k}\ .
\end{aligned}
$$

List of publications reproduced in the thesis

We collect in the following pages all the relevant information about the publications used in the thesis. We have reproduced part of their content, especially the most technical one, in the appropriate parts of the referred chapters.

T-duality and high-derivative gravity theories: the BTZ black hole/string paradigm, *Journal of High Energy Physics* **2018**, 142 (2018)

Authors

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PhD Student Contribution

Active participation in all the calculations presented in the paper, as well as in the discussions and meetings.

Chapter of the thesis in which it is used: 3

Journal and Article Information

Journal name: Journal of High Energy Physics
Publisher: Springer
ISSN: 1029-8479 (electronic)
Year of publication: 2018
DOI: `https://doi.org/10.1007/JHEP06(2018)142`
Impact factor in 2018: 5.833 – JCR data
Position in Physics, Particles & Fields in 2018: 5/29 (Q1) – JCR data

T-duality equivalences beyond string theory, *Journal of High Energy Physics* **2019**, 82 (2019)

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Chapters of the thesis in which it is used: 1, 2 & Appendix A

Journal and Article Information

Journal name: Journal of High Energy Physics
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ISSN: 1029-8479 (electronic)
Year of publication: 2019
DOI: `https://doi.org/10.1007/JHEP05(2019)082`
Impact factor in 2019: 5.875 – JCR data
Position in Physics, Particles & Fields in 2019: 4/29 (Q1) – JCR data

The first α'-correction to homogeneous Yang-Baxter deformations using $O(d,d)$ **2020**, 103 (2020)

Authors

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Active participation in all the calculations related to T-duality and TsT transformations in the paper. Test of the results in some examples.

Chapter of the thesis in which it is used: 1

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Journal name: Journal of High Energy Physics

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Year of publication: 2020

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Impact factor in 2020: 5.810 – JCR data

Position in Physics, Particles & Fields in 2020: 5/29 (Q1) – JCR data

Entanglement entropy in cubic gravitational theories, *Journal of High Energy Physics* **2021**, 186 (2021)

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Journal and Article Information

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Position in Physics, Particles & Fields in 2020: 5/29 (Q1) – JCR data

Holographic entanglement entropy for perturbative higher-curvature gravities, *Journal of High Energy Physics* **2021**, 145 (2021)

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PhD Student Contribution

Active participation in all the calculations presented in the paper, as well as in the discussions and meetings.

Chapter of the thesis in which it is used: 4, 5, Appendix B & Appendix C

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Resumo en galego

Esta tese está centrada no estudo teórico da entropía nas teorías gravitacionais que inclúen no seu lagranxiano correccións de orde superior na curvatura. A entropía gravitacional é un concepto que xorde ao abeiro da física dos buracos negros, tras os resultados a mediados da década dos 70 que indicaron que estes obxectos posúen unha entropía proporcional á área do seu horizonte de eventos [14, 15]. Desde o primeiro momento, este resultado foi recoñecido como un dos poucos indicios que posuímos de cara a construír unha teoría cuántica da gravidade, que debera ser quen de reproducir esa entropía proporcional á área a partir dun reconto de microestados gravitacionais. Numerosos desenvolvementos no eido da física teórica de altas enerxías foron desde entón iluminados por esta noción da entropía gravitacional. O máis salientable é quizais a proposta da holografía [18, 19]: unha teoría cuántica da gravidade nun certo número de dimensións debera posuír unha descrición en termos dunha teoría non gravitacional nunha dimensión menos, o que implementaría de xeito natural a proporcionalidade entre a entropía dunha certa rexión e a área da superficie que a envolve. A correspondencia AdS/CFT proposta por Juan Maldacena [21] é a realización máis concreta e exitosa ata o momento deste principio: esencialmente afirma que unha certa forma de teoría cuántica da gravidade (en concreto, unha teoría de cordas) nun espazo asintóticamente AdS pode ser descrita en termos dunha teoría cuántica de campos conforme (CFT) na fronteira do devandito espazo. De entre os numerosos resultados que se foron producindo nas últimas décadas no marco da correspondencia AdS/CFT, especialmente salientable para a presente tese son os que teñen que ver coa entropía de entrelazamento holográfica. É posible calcular a entropía de entrelazamento dunha certa rexión da CFT a partir da área dunha superficie asociada no volume interior gravitacional, vía a coñecida como proposta de Ryu e Takayanagi [32]. Esta nova noción de entropía gravitacional, novamente vencellada a unha superficie e á súa área, non é totalmente independente da anteriormente comentada para os buracos negros. O propio Maldacena, xunto con Lewkowycz [34], amosou que, baixo a hipótese de que a gravidade se comporta holograficamente, ambos os dous resultados proveñen dunha noción única de entropía gravitacional, axeitadamente definida en termos da entropía cuántica de von Neumann na teoría dual non gravitacional.

Os resultados concretos desta tese divídense en dúas partes, cada unha delas centrada no estudo dunha forma de entropía (de buracos negros, ou de entrelazamento holográfica) en teorías con correccións de orde superior na curvatura. A proporcionalidade entre entropía gravitacional e área é unicamente válida cando consideramos a nosa teoría xeométrica da gravidade gobernada pola acción de Einstein-Hilbert. Se incluímos no lagranxiano contraccións de orde superior da curvatura, o funcional que debemos avaliar na superficie que calcula a entropía recibe correccións, o que pode facer cambiar as súas propiedades. Boa parte do traballo da presente tese céntrase en estudar estas correccións e os cambios que producen, tanto no contexto da entropía de buracos negros como no da entropía de entrelazamento holográfica. Porén, convén antes de entrar no detalle destes resultados xustificar o interese de considerar correccións de orde superior na curvatura nas nosas teorías. Desde un punto de vista fundamental, adoptando a visión Wilsoniana das teorías cuánticas de campos que se impuxo durante a segunda metade do século XX, a acción de Einstein-Hilbert debera ser só unha aproximación válida a baixas enerxías na que implicitamente estamos a descartar vía integración os modos con enerxía por riba dunha certa escala límite. Estes modos presentes na teoría que describe a gravidade no ultravioleta manifféstanse a baixas enerxías producindo tamén correccións de orde superior na curvatura, suprimidas respecto da acción de Einstein-Hilbert por unha certa escala de enerxías. Así pois, a presenza das correccións é esperada, e estudar as súas consecuencias para os cálculos de entropía gravitacional, sendo este un concepto fortemente vencellado á descrición microscópica no ultravioleta da gravidade, é un problema natural no contexto da renormalización Wilsoniana. O marco no que de xeito máis directo pode ser observada a presenza destas correccións de orde superior na curvatura é o da teoría de cordas, que predí accións a baixas enerxías que efectivamente posúen esas correccións. Na primeira parte da tese, as teorías efectivas a baixas enerxías motivadas desde a teoría de cordas serán o noso campo de traballo para estudar a entropía dos buracos negros.

Podemos aducir tamén unha segunda razón para o interese nas teorías con correccións de orde superior na curvatura, desta volta cunha motivación máis procedente da holografía e da correspondencia AdS/CFT. Nesta correspondencia, a dualidade relaciona unha teoría de campos no réxime de acoplamento forte, co límite semiclásico da teoría cuántica da gravidade nun espazo AdS. Este límite semiclásico pode incluír correccións de orde superior na curvatura (de feito, como xa se mencionou, na teoría de cordas aparecen), e as diferentes accións que gobernan a dinámica gravitacional manifféstanse do lado da CFT como teorías de campos conformes con diferentes propiedades. O caso máis coñecido deste feito é probablemente o da cota KSS e a súa violación: mentres que considerando a gravidade de Einstein no lado AdS podemos obter unha cota inferior para o cociente entre a viscosidade de cizalladura e a densidade de entropía na CFT [44], esta cota non é respectada se do lado gravitacional incluímos correccións de orde superior na curvatura, como o termo cadrático de Gauss-Bonnet [45, 46]. Indo na dirección oposta, a obtención de resultados válidos para as CFTs duais a un conxunto amplo de teorías gravitacionais con correccións de orde superior na curvatura permite ás veces descubrir relacións universalmente válidas nas teorías de campos conformes, que logo poden ser verificadas con métodos non holográficos [176]. Todo isto motiva o estudo das teorías gravitacionais con correccións de orde superior na curvatura no contexto da correspondencia AdS/CFT cun espírito próximo ao fenomenolóxico, xa que son unha ferramenta útil para acceder a diferentes teorías de campos conformes. Esta é a motivación fundamental para a segunda parte da tese, na que estudamos a entropía de entrelazamento holográfica cando a acción gravitacional contén termos de orde superior na curvatura.

Entrando no detalle da primeira parte da tese, o obxectivo da mesma é estudar a entropía de buracos negros nunha familia de teorías gravitacionais motivada polas accións efectivas a baixas enerxías da teoría de cordas. Unha propiedade fundamental desta teoría é a coñecida como invariancia baixo T-dualidade, que aparece como consecuencia da simetría entre os modos de momento e os modos de enroscamento que posúen as cordas cando se cuantizan en espazos con dimensións compactas. Desde o punto de vista da acción a baixas enerxías que goberna a dinámica do espazotempo no que poden vivir as cordas, esta simetría manifístase como unha equivalencia entre dúas solucións nas que o tamaño das dimensións compactas vese invertido. A transformación concreta entre unha solución e a súa dual vén dada polas coñecidas como regras de Buscher [65]. Un pode preguntarse, desde o punto de vista da teoría efectiva a baixas enerxías, se a existencia da T-dualidade como simetría é unha restrición suficientemente forte como para ditar a forma da teoría, obrigando a que sexa unha das que proceden das distintas teorías de cordas, incluso cando se inclúen nela correccións de orde superior na curvatura (neste contexto, habitualmente chamadas correccións α'). A resposta a esta pregunta incluíndo correccións de primeira orde foi proporcionada por Marqués e Núñez no contexto da coñecida como *Double Field Theory* (DFT) [48]: a primeira orde, existe unha familia de teorías con dous parámetros libres que respectan a versión en termos de solucións da T-dualidade. Para certos valores dos parámetros, estas teorías son as proporcionadas polas teorías de cordas no límite de baixas enerxías, pero non para todos. Así pois, a T-dualidade é unha simetría que ten sentido máis alá dos límites das teorías de cordas, canto menos desde o punto de vista das accións gravitacionais a baixas enerxías. Se consideramos buracos negros nestas teorías, é posible preguntar que ocorre coa súa entropía baixo o efecto dunha transformación de T-dualidade. En primeiro lugar, levan as transformacións de T-dualidade solucións de buracos negros en solucións de buracos negros? Permanece neste proceso invariante a entropía (e a temperatura) dos buracos negros? Para aquelas teorías que proveñen da teoría de cordas, semella evidente que a resposta debe ser si: a T-dualidade é unha equivalencia física total, consecuencia das propiedades das cordas cuánticas cando existen direccións compactas. A baixas enerxías esta equivalencia debe manterse, e en particular as propiedades das solucións (entropía, temperatura, etc.) non deberan verse afectadas. Sen embargo, para aqueles valores dos parámetros da familia de teorías de Marqués e Núñez que non se corresponden con teorías de cordas a baixas enerxías, a resposta semella menos clara. Pode a T-dualidade proporcionar unha equivalencia física total entre solucións, ou hai certas magnitudes, en particular aquelas relacionadas coa termodinámica de buracos negros (que deberan ser sensibles ás propiedades ultravioletas da teoría), que poden sufrir cambios baixo a transformación de T-dualidade cando non nos atopamos nos valores dos parámetros procedentes das teorías de cordas?

A primeira parte da tese dá resposta a esta pregunta: polo menos a primeira orde e no que se refire ás propiedades termodinámicas dos buracos negros, a T-dualidade proporciona unha equivalencia física total. Con independencia de que os valores dos parámetros sexan ou non os das teorías de cordas, a entropía e a temperatura dos buracos negros non se ven afectadas por unha transformación de T-dualidade. Para amosar isto rigorosamente, o capítulo 1 da tese discute a construción da familia de teorías de Marqués e Núñez, facendo énfase nas súas simetrías (esenciais para obter a entropía dos buracos negros) e na forma que adoptan as regras de T-dualidade cando se inclúen as correccións perturbativas de primeira orde na acción. Isto é particularmente relevante dado que sempre podemos permitir redefinicións perturbativas dos campos fundamentais da teoría, que afectan á forma das regras de T-dualidade. O capítulo conclúe facendo un compendio das distintas regras segundo a redefinición dos campos que fagamos, o que serve tamén como referencia para traballos futuros nos que estas regras de T-dualidade con correccións de primeira orde precisen ser empregadas. O capítulo 2 estuda entón a forma da entropía dos buracos negros na familia de teorías, e a súa invariancia xeral baixo T-dualidade. O proceso de construción da integral que, avaliada sobre o horizonte dos buracos negros, proporciona a súa entropía, é tecnicamente complexo e require da utilización dos métodos propostos por Wald [51, 84], que interpretan a entropía dos buracos negros como unha certa carga de Noether en teorías con invariancia baixo difeomorfismos. A totalidade das simetrías da familia de teorías debe ser considerada neste proceso, e unha lixeira xeneralización do método de Wald permite entón obter a forma do funcional de entropía para buracos negros. O apéndice A apoia esta construción, quedando relegadas a ese lugar as partes máis técnicas da derivación. Unha vez a forma da entropía de buracos negros é coñecida, estúdase nese mesmo capítulo 2 a invariancia da mesma baixo T-dualidade. Para facer isto en xeral, trabállase cun horizonte de Killing bifurcado arbitrario, introducindo un sistema de coordenadas adaptado ao mesmo e derivando as propiedades de transformación dos distintos campos da teoría. Cando estas son tomadas en consideración, pode probarse a invariancia baixo T-dualidade da entropía de buracos negros para calquera valor dos parámetros da familia de teorías, así como tamén da temperatura. O último capítulo desta primeira parte, o terceiro da tese, constitúe entón un exemplo concreto dos resultados xerais e abstractos do capítulo previo. Considerando a solución BTZ tridimensional [97, 99], válida para calquera valor dos parámetros da familia de teorías, aplicamos explicitamente as transformacións de T-dualidade corrixidas para obter o dual, e confirmamos que é unha solución cun horizonte de eventos para o que a temperatura e a entropía son as mesmas que as da solución BTZ orixinal. Como resultado colateral da verificación nun caso concreto dos argumentos abstractos do capítulo previo, presentamos explicitamente a solución dual ao BTZ incluíndo correccións de primeira orde. Esta é unha versión corrixida da corda negra tridimensional xa coñecida nas teorías efectivas a baixas enerxías procedentes das teorías de cordas, amplamente estudada no pasado [100].

A segunda parte da tese desenvólvese no contexto da correspondencia AdS/CFT, e está centrada no estudo do funcional de entropía de entrelazamento holográfica cando na teoría gravitacional aparecen contraccións dos tensores de curvatura de orde arbitraria. O estudo deste problema foi iniciado polos traballos de Xi Dong e Joan Camps [52, 53], que propuxeron unha forma xeral para o funcional cando no lagranxiano aparecen contraccións arbitrarias do tensor de Riemann. Estes traballos estaban fortemente baseados na construción de Lewkowycz e Maldacena, que xustificou a validez da proposta de Ryu e Takayanagi. Con todo, a presenza de termos de orde superior na curvatura implicaba unha nova sutileza que foi pasada por alto nun primeiro momento. A construción de Lewkowycz e Maldacena require dunha regularización da métrica nunha singularidade cónica que non é única, e para cada teoría gravitacional a forma correcta debe vir ditada polas ecuacións de movemento. Esta ambigüidade foi coñecida como o "splitting problem", e posta de manifesto principalmente nos traballos de Miao e Camps [115–117]. A elevada complexidade das ecuacións de movemento nas teorías con correccións de orde superior na curvatura fai que non se coñeza a correcta resolución do problema en ningunha teoría que non sexa gravidade de Einstein ou as variantes de orde superior que posúen certas propiedades moi particulares, como teorías de Lovelock ou $f(R)$. Con todo, un pode impoñer unha condición que simplifica o tratamento. Se se traballa no réxime no que as correccións de orde superior na curvatura son perturbativas, a solución para o “splitting problem” é a mesma que se ten para gravidade de Einstein, posto que as modificacións producidas pola diferenza nas ecuacións do movemento son polo menos cadráticas na expansión perturbativa do funcional de entropía, e polo tanto irrelevantes a primeira orde. Nesta tese a visión adoptada é esta: traballaremos a nivel perturbativo, de xeito que o “splitting problem” pode ser ignorado. Aínda nesa situación, o procedemento de obtención do funcional de entropía de entrelazamento é complexo, e involucra unha expansión en compoñentes dos tensores de curvatura que é dificilmente interpretable e implementable en sistemas computacionais. Isto fai que, con anterioridade aos traballos da presente tese, poucos funcionais concretos fosen coñecidos máis alá das correccións cadráticas na acción.

O primeiro capítulo da segunda parte, o 4, ten como finalidade remediar esta situación. Tras unha breve exposición da construción que proporciona o funcional de entropía de entrelazamento en teorías con correccións de orde superior na curvatura, desenvólvese un novo método para obtelo. Nesta proposta, a expansión en termos das distintas compoñentes dos tensores de curvatura faise de xeito automático mediante uns operadores diferenciais axeitadamente definidos. A proba técnica da validez desta nova reescritura do funcional queda relegada ao apéndice B, pero no propio capítulo discútense as vantaxes que presenta esta forma en termos dos operadores. Por unha banda, permite comprender mellor a estrutura do funcional para unha teoría xeral, e inclúense así diversas discusións ao respecto dependendo do número de tensores de Riemann presentes no lagranxiano. Por outra, a implementación nun software de cálculo tensorial é moito máis doada para a nova forma do funcional. Isto permite obter eficientemente os funcionais correspondentes a teorías cadráticas, cúbicas e cuárticas, que son explicitamente presentados no propio capítulo 4, compendiando tamén no apéndice C as expresións máis longas correspondentes ás teorías cúbicas e cuárticas. En resumo, este capítulo aspira a ser unha referencia xeral para o cálculo de funcionais de entropía de entrelazamento en teorías de orde superior: posúe a exposición xeral dun método eficiente para a tarefa, inclúe discusións xerais sobre o mesmo, e facilita tamén resultados concretos para teorías con contraccións de ata catro tensores de curvatura. Todos os resultados concretos asumen que as correccións de orde superior son perturbativas, pero tamén se discute que, no caso de traballar cunha solución diferente do “splitting problem”, o método debera ser adaptable, canto menos na súa idea fundamental de empregar operadores diferenciais para realizar a expansión en compoñentes requirida no funcional.

Finalmente, o quinto e último capítulo da segunda parte da tese constitúe unha aplicación concreta dos resultados do anterior, coa mirada posta en obter propiedades relevantes das CFTs duais ás teorías con correccións de orde superior na curvatura mediante as entropías de entrelazamento de diversas rexións na teoría de campos. Tras unha introdución xeral, e considerando sempre a solución AdS pura da teoría de gravidade (dual ao baleiro da CFT) con correccións ata orde cúbica, preséntanse os resultados para os termos universais da entropía de entrelazamento de rexións na CFT con forma de esferas, bandas, (hiper)cilindros, e cuñas. Estes termos universais son cantidades ben definidas (independentes do regulador no ultravioleta) que aparecen na entropía de entrelazamento das diversas rexións consideradas na CFT, e proporcionan por tanto parámetros que caracterizan as teorías. O caso máis coñecido é o das esferas, para as que en dimensión par da CFT o termo universal está relacionado cun dos coeficientes da anomalía conforme, mentres que en dimensión impar vén dado pola enerxía libre da CFT nunha variedade (euclídea) esférica. Noutras xeometrías ás veces podemos identificar algún dos termos universais (caso dos cilindros en dimensión par, que poden proporcionar todos os coeficientes da anomalía conforme), mais ás veces obtemos cantidades independentes intrínsecas da CFT non relacionadas con outras xa coñecidas (caso das bandas). Particularmente interesantes son os resultados correspondentes ás cuñas dunha CFT tridimensional. A entropía de entrelazamento nestas xeometrías vén caracterizada por unha función que depende do ángulo da cuña, que é coñecida para certas teorías, como un escalar ou un fermión libres. Holograficamente, o resultado de gravidade Einstein era tamén coñecido, e as correccións cadráticas en curvatura non producían unha nova forma funcional dependendo do ángulo de apertura da cuña. A inclusión de termos cúbicos na acción, empregando o correspondente funcional, permite sen embargo obter unha nova función para a cuña, como amosamos na parte final deste capítulo 5. Unha vez máis, isto proporciona unha proba da relevancia e utilidade das teorías con correccións de orde superior na curvatura no contexto da holografía e a correspondencia AdS/CFT, especialmente cando se consideran como mecanismo de estudo fenomenolóxico das CFTs. Conxecturas previas, que apuntaban a que o resultado para gravidade Einstein constituía unha cota inferior xeral para as funcións da cuña, poden amosarse falsas unha vez se inclúen correccións cúbicas na teoría gravitatoria.

A tese remata cun breve capítulo 6 no que damos unha visión xeral dos resultados obtidos e algunhas conclusións, que pretenden complementar ás máis específicas aportadas ao final de cada un dos capítulos.

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